Extracting white noise statistics in GPS coordinate time series

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***Abstract*— The noise in GPS coordinate time series is known to follow a power-law noise model with different components (white noise, flicker noise, random-walk). This work proposes an algorithm to estimate the white noise statistics, through the decomposition of the GPS coordinate time series into a sequence of sub-time series using the Empirical Mode Decomposition algorithm. The proposed algorithm estimates the Hurst parameter for each sub time series, then selects the sub time series related to the white noise based on the Hurst parameter criterion. Both simulated GPS coordinate time series and real data are employed to test this new method, results are compared to the standard (CATS software) Maximum Likelihood (ML) estimator approach. The results demonstrate that this proposed algorithm has very low computational complexity and can be more than one hundred times faster than the CATS ML method, at the cost of a moderate increase of the uncertainty ($\sim 5\%$) of the white noise amplitude. Reliable white noise statistics are useful for a range of applications including improving the filtering of GPS time series, checking the validity of estimated coseismic offsets and estimating unbiased uncertainties of site velocities. The low complexity and computational efficiency of the algorithm can greatly speed up the processing of geodetic time series.**

***Index Terms*—Empirical Mode Decomposition, white noise amplitude, Hurst parameter, fractional-Brownian motion, power-law noise, GPS coordinates.**

I. Introduction

THe application of Global Navigation Satellite System (GNSS) observations for monitoring geophysical phenomena such as earthquakes and tectonic movements requires also understanding the long term coordinate time series error spectum. This paper investigates different methods to best fit the noise contained within GPS station coordinate time series. The general method used is to fit and remove a linear trend to the coordinate time series and then model the noise characteristics of the residuals (e.g. [10], [19]). It is crucial to know the statistics of the different noise components for applications such as checking the significance of estimated earthquake coseismic offsets and/or tectonic velocities from noisy GPS time series (e.g. [10] and [19]). Several studies have shown that the error spectrum of geodetic GNSS time series is best characterised by a stochastic process following a power-law (e.g. [19]) as:

$$S(f) \simeq 1/f^{\alpha}, \qquad \alpha \in [0, 2] \tag{1}$$

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with the power spectrum ($S(f)$), and α the spectral index. Following this model, researchers demonstrated that the noise is coloured with mainly three components: white noise, flicker noise and random walk [9]. White noise is independent of frequency, and is generally associated with hardware noise or measurement errors. Clearly, white noise contains little or no geophysical information. However, it is useful to have a good knowledge of the white noise statistics to enable efficient filtering. For example, GPS time series can be filtered using a Kalman filter that require a priori knowledge of the noise statistics [5]. Recently, authors in [12] developed a least mean square adaptive filter to smooth GPS time series based on a complex noise model. In addition, knowledge of the noise statistics would help to exclude rather noisy GPS time series when processing measurements from a global network of stations to estimate geophysical parameters such as the coseismic offsets associated with earthquakes [8], and offsets arising from instrument upgrades and changes [17]. A power-law noise model means that $S(f)$ is not flat but is governed by long-range dependencies. If the probability density function of the noise is Gaussian or has a different density function with a finite value of variance, its fractal properties can be described by the Hurst parameter (H). In 1968, [11] defined the fractional Brownian motion (fBm) model where the H is studied. In the case of $H < 0.5$, the process behaves as a Gaussian variable; if $H > 0.5$ the process exhibits long-range dependence; while the case of $H = 0.5$ corresponds to a pure Brownian motion (white noise).
From [16], H is directly connected with α by the relation:

$$\alpha = 2H - 1, \qquad \alpha \leq 2 \tag{2}$$

With this definition, flicker noise corresponds to $\alpha = 1$ or $H = 1$, the random walk to $\alpha = 2$ or $H = 3/2$, and white noise is related to $\alpha = 0$ ($H = 0.5$). Thus, the random walk and the flicker noise are classified as long-term dependency phenomena. It is, however, difficult to look directly into the GPS time series to characterize the various fractal properties because the different noise components are correlated.

In this study, we apply the fBm model combined with the Empirical Mode Decomposition (EMD) algorithm onto each GPS coordinate time series. First a GPS time series is decomposed into sub-time series using the EMD. Then the Hurst parameter is estimated for each sub-time series. Finally the sub-time series with $H \leq 0.5$ are selected to extract the statistics of the white noise. The reliability of the proposed algorithm is evaluated and performance is compared in term of accuracy and computation time with the Maximum-Likelihood Estimation (MLE) via the use of the CATS software developed by [19]. Note that the noise statistics in GPS coordinate time series with MLE was successfully estimated in the early work of [10]. For the past decade, CATS has been internationally used in various geophysical studies (e.g. [13]). More recently, [2] underlined the long processing time and improved the software by modifying the algorithm.

The next section reviews the EMD and details the proposed algorithm. Section III describes the various testbeds, shows the simulation results and provides some analysis. Finally, some concluding remarks are drawn.

II. The Empirical Mode Decomposition and the Proposed Algorithm

The EMD method consists of decomposing a time series, $x(t)$, into sub-series, also called Intrinsic Mode Function (IMF). The decomposition of $x(t)$ is done via the study of consecutive local extrema (i.e. two minima in the time interval $[t_1, t_2]$). The general formula is given by:

$$x(t) = \sum_{i=1}^{K} d_i(t) + m_K(t) \tag{3}$$

where d_i is the i-th IMF and m_K is the residual after K decomposition. Each IMF is the result of applying a different band-pass filter with certain cutoff frequencies. [7] explained comprehensively the EMD. Several studies have already shown the possibility of extracting white noise statistics from noisy signal with this method (e.g. [3]). In [4], the authors modeled the EMD method as a dyadic filter bank resembling those involved in wavelet decompositions. They also pointed out the evolution of the Hurst parameter (H) in the IMFs when decomposing a fractional Gaussian noise (resulting in H spanning the whole interval $[0, 1]$). According to [4], the pseudo code of the EMD algorithm is formulated as:

- Step 1: find all the maxima and minima points. Fit the upper envelope according to the maxima points, and the lower envelope according to the minima points.
- Step 2: compute the mean value ($m(t)$) of the upper and lower envelopes.
- Step 3: compute the difference value ($d(t)$) between the time series $x(t)$ and the mean value $m(t)$.
- Step 4: iterate on the residual $m(t)$.

The iteration in the last step of the EMD stops when $d(t)$ is considered as zero-mean according to some stopping criterion. This stopping criterion will influence the number of IMFs extracted from the original time series, with the first IMFs corresponding to the high frequencies. The number of the IMFs is generally linked to the level of noise in the time series [4]. Next, the fBm model is exploited to select the appropriate IMFs based on the estimated H parameters to calculate the statistics of the white noise. The pseudo code of the proposed algorithm can be written as:

- Step 1: Decompose a nominated time series $x(t)$ into K IMFs.
- Step 2: Estimate the Hurst parameter for each IMF.
- Step 3: Sum the IMFs whose H values are within $[0, 0.5]$, resulting in $s_1(t)$. Also sum the IMFs whose H values are in the range of $[0, 0.6]$, resulting in $s_2(t)$.
- Step 4: Calculate the statistics of the white noise as $Amp = std(s_1(t))$ and $Un_w = |std(s_2(t)) - std(s_1(t))|$.

where Amp and Un_w are the estimated amplitude of the white noise (or standard deviation) and the uncertainty of the amplitude of the white noise, respectively. Note that std means the standard deviation function. As for Step 2, a range of estimators as described in [14] can be used to estimate H. However, it was observed that the method developed by [12], based on the self similarity properties of the fractional-Brownian noise and the boxed periodogram based method in [14], are best suited to deal with this problem. In addition, it has been shown that many methods for calculating the H parameter suffer from a significant error margin (up to 0.1). Thus in Step 4, the uncertainty of the amplitude of the white noise is calculated using the IMFs with $H < 0.6$ which takes into account the fact that some white noise information may be contained in the IMFs with H values larger than 0.5.

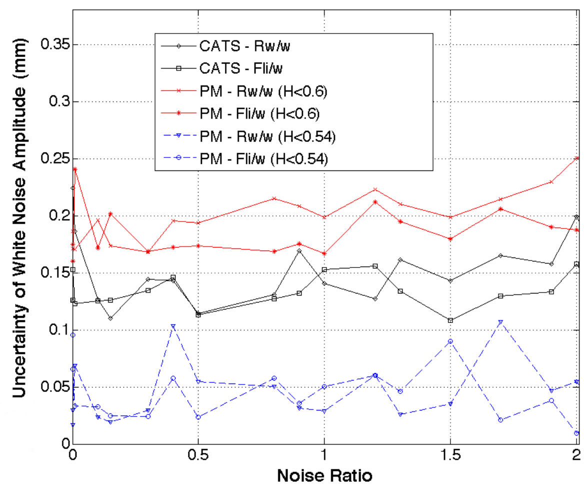

Fig. 1. Uncertainty of the white noise amplitude estimated with the proposed method (Un_w) with different criteria ($H < 0.6$ and $H < 0.54$) and with CATS averaged on 20 simulations versus the noise ratio [Random Walk over white noise (Rw/W), Flicker noise over white noise (Fli/W)].

III. Processing and Results

The proposed algorithm is first applied to simulated time series, which contain 4 different components: Gaussian noise, a seasonal component, flicker noise and a random walk component as described in [19] or [18]. Note that the seasonal component models the signals found in geodetic time series at annual and semi-annual frequencies. The amplitude varies for each simulation, with the seasonal component equal to half of the amplitude of the white noise. However, the information on the simulated seasonal component are not displayed in the following results as it has very little influence on the general results given by the two methods. The random walk (or Brownian motion) is simulated using the MATLAB function $cumsum$ (integration of the white noise). The flicker noise is simulated following the derivative method used in [15]. Three sets of simulations are performed with the parameters described in Table I. Table II shows the results (in mm) of the estimated white noise statistics averaged over 20 simulated time series with a variable number of epochs. The amplitude of each noise component varies for each simulation according

TABLE II

RESULTS (MM) OF THE PROPOSED METHOD (PM) AND CATS ALGORITHM APPLIED TO SIMULATED TIME SERIES ($[Amp]$ ESTIMATED WHITE NOISE AMPLITUDE, $[error]$ ABSOLUTE ERROR OF THE ESTIMATED WHITE NOISE AMPLITUDE, $[Un_w]$ UNCERTAINTY OF THE ESTIMATED WHITE NOISE AMPLITUDE)

Simulations	Method	Amp		Error		Un_w	
		Mean	Std	Mean	Std	Mean	Std
Case I	**PM**	5.4	± 1.2	0.1	± 2.0	0.1	± 0.1
	CATS($\alpha = 1$)	5.2	± 1.1	0.3	± 0.2	0.2	± 0.1
	CATS (α =2)	5.5	± 1.2	0.2	± 0.1	0.1	± 0.0
Case II	**PM**	5.2	± 1.2	0.1	± 0.1	0.1	± 0.2
	CATS($\alpha = 1$)	5.0	± 1.2	0.2	± 0.1	0.1	± 0.0
	CATS($\alpha = 2$)	5.3	± 1.2	0.2	± 0.1	0.1	± 0.0
Case III	**PM**	5.7	± 1.1	0.1	± 0.1	0.1	± 0.2
	CATS($\alpha = 1$)	5.0	± 1.0	0.2	± 0.1	0.1	± 0.0
	CATS($\alpha = 2$)	5.8	± 1.1	0.2	± 0.1	0.1	± 0.0

TABLE I

SIMULATION PARAMETERS TO TEST THE TWO ALGORITHMS ($[white]$ WHITE NOISE, $[RW]$ RANDOM-WALK, $[flicker]$ FLICKER NOISE, $[Amp]$ AMPLITUDE OF THE NOISE COMPONENTS, $[Std]$ STANDARD DEVIATION OF THE SIMULATED AMPLITUDE OF THE NOISE COMPONENTS, $[N.S]$ NUMBER OF SIMULATIONS, $[Length]$ NUMBER OF EPOCHS.

Simulation Parameters	Type		Amp	Std	N.S.	Length
Case I	white	(mm)	5.4	± 1.2		
	flicker	$(mm/yr^{1/4})$	4.0	± 1.8	20	800
	RW	$(mm/yr^{1/2})$	4.0	± 1.8		
Case II	white	(mm)	5.1	± 1.2		
	flicker	$(mm/yr^{1/4})$	3.0	± 1.4	20	1000
	RW	$(mm/yr^{1/2})$	3.0	± 1.4		
Case III	white	(mm)	5.6	± 1.1		
	flicker	$(mm/yr^{1/4})$	3.8	± 1.8	20	1500
	RW	$(mm/yr^{1/2})$	3.8	± 1.8		

to the values in Table I. Note that $Error$ in Table II is the absolute error between the true amplitude and estimated value of the white noise component. Overall, the simulated results show that the proposed method performs closely to the CATS software in all three scenarios in terms of estimated amplitude error. However, the standard deviation of the uncertainty (Un_w) is sometime larger than the mean value for the proposed method. In [20], the author defines CATS' uncertainties using the Fisher information matrix. This may justify the low values as directly related to the asymptotic Cramer-Rao lower bound [5]. In order to investigate this result, we simulate 20 time series (1000 epochs) with the three noise components (but with Flicker noise or Random Walk amplitude kept constant at 0.003 $mm/yr^{1/2}$ and 0.003 $mm/yr^{1/4}$ respectively) for each noise ratio. The results in Figure 1 are the averaged uncertainties over the 20 simulations. We mentioned earlier that the criterion can be adjusted depending on how accurate is the algorithm used to estimate the Hurst parameter. As an example, Figure 1 also displays the uncertainties of the proposed method when using $H < 0.54$. The mean uncertainty values with the proposed method are higher than CATS' results when selecting the threshold $H < 0.6$, whereas the results with our method using the modified criterion ($H < 0.54$) are smaller than CATS. $H < 0.6$ may be too large criterion which may explain the large uncertainties.

We processed GPS observations from a globally distributed network of over 120 sites following the procedure detailed in [18]. We used the $VMF1$ mapping function

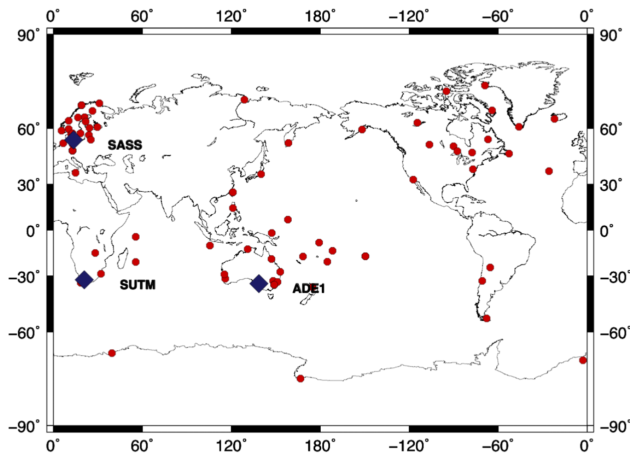

Fig. 2. The 78 GPS Stations used in this study.

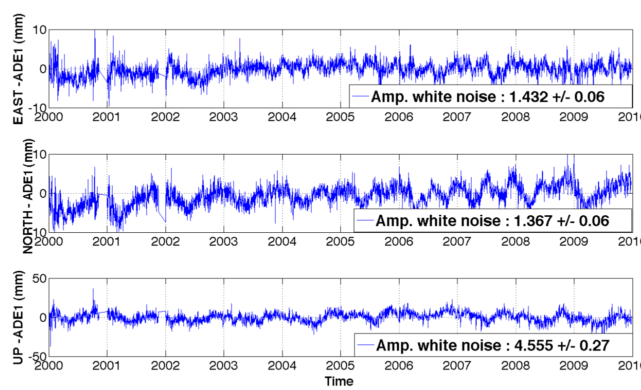

Fig. 3. Coordinates of the GPS station $ADE1$ together with the estimated amplitude of the white noise (in mm) by the proposed algorithm after correcting the data from the tectonic rate.

and zenith hydrostatic delay from the $ECMWF$ numerical model [1]. The GPS observations were analyzed using the $GAMIT/GLOBK$ software [6] as three, interwoven global subnetworks of 40 sites, each with at least five common sites between the subnetworks. For each solution, the loosely constrained daily subnetworks were combined in a second step where the terrestrial reference frame was defined by performing a six-parameter transformation of the coordinates of 30 sites onto the $ITRF2008$ [18]. The steps undertaken to realize the reference frame were identical for all solutions. Finally, 78 stations were selected with a recording data period between 3 and 10 years (between the years 2000 and 2010).

TABLE III
ESTIMATED STATISTICS OF THE WHITE NOISE AMPLITUDE [Amp] AND UNCERTAINTY [Un_w] WHEN USING CATS OR THE PROPOSED METHOD ($P.M.$) ON REAL GPS TIME SERIES (IN MM)

GPS Station			**CATS** $\alpha = 1$	$\alpha = 2$	**P.M.**
ADE1	East	Amp	1.170	1.585	1.432
		Un_w	0.053	0.023	0.062
	North	Amp	1.080	1.401	1.367
		Un_w	0.037	0.021	0.063
	Up	Amp	2.813	4.019	4.555
		Un_w	0.122	0.0587	0.271
SUTM	East	Amp	1.938	2.301	1.850
		Un_w	0.065	0.040	0.066
	North	Amp	1.556	1.967	1.898
		Un_w	0.062	0.036	0.105
	Up	Amp	3.484	4.094	4.140
		Un_w	0.111	0.073	0.240
SASS	East	Amp	0.741	0.965	0.980
		Un_w	0.031	0.016	~ 0
	North	Amp	0.647	0.907	0.933
		Un_w	0.032	0.016	0.180
	Up	Amp	2.904	3.180	3.668
		Un_w	0.417	0.075	0.270

The distribution of sites analyzed is shown in Figure 2.

We applied both algorithms to the real GPS coordinate time series. The CATS software was used in a mode where we set the order of the dominant noise equal either to 1 (flicker noise) or 2 (random walk). Note that the mode with free order of the power-law noise was preliminary tested and gave no comparable results in our simulations. We removed the linear tectonic drift using the $polyfit$ function in MATLAB before processing the time series with either the proposed algorithm or CATS in order to have the same residual error. Figure 3 shows the time series for $ADE1$ and it is typical of the other time series. Results are displayed in Table III for the North, East and Up coordinates (in mm) for two stations. The results show that the proposed method is estimating the white noise well compared to the results given by CATS, in general. That is, the amplitude of the white noise lies in a range closed to the results calculated with CATS. For example in the case of $SUTM$, the estimated white noise amplitude with the proposed method is 1.898 mm for the North coordinate and the CATS results showed the amplitude between 1.556 mm and 1.967 mm. However, the uncertainty of the estimated white noise amplitude is either higher or very small for the proposed algorithm. As previously stated, this occurs because the estimation of the Hurst parameter suffers from a large standard deviation (e.g. [14], [12]). Thus, a very small uncertainty means that the selected additional IMFs do not introduce more white noise amplitude. One could then choose a bigger criterion in Step 3 of the proposed algorithm (i.e. H in $[0, 0.7]$).

In order to demonstrate the flexibility of this algorithm, the processing time (in second) is displayed in Table III for each time series shown in Table III. Note the results displayed in Table III for each algorithm are averaged for one coordinate. The results show that the proposed algorithm performs much faster than CATS. The processing times indicate that CATS is an $O(n^3)$ process (the modified version of [2] is an $O(n \log n)$), whereas the proposed method is an $O(n)$ process. CATS is using the Nelder-Mead simplex algorithm to minimize a cost function based on the log-likelihood with the parameters corresponding to the amplitude of the various noises. As such, the work in [19] shows that the estimated parameters converge to the minimum variance unbiased estimators. The drawback is the long converging time and the need for long GPS time series in order to calculate a reliable estimate (i.e. typically 1.5 years when using GPS daily solution). Here, the performances of the proposed algorithm are independent of the length of the time series because the process acts as a filter. Nevertheless, we also acknowledge that CATS processing time is the total time when calculating the amplitude and uncertainties for the white noise and other noise component (such as flicker noise and/or random-walk).

TABLE IV
PROCESSING TIME FOR SOME GPS TIME SERIES (S) AVERAGED FOR ONE COORDINATE FOR THE PROPOSED METHOD ($P.M.$) AND THE SOFTWARE CATS

GPS Station	**CATS** $\alpha = 1$	$\alpha = 2$	**P.M.**	**Length**
ADE1	966.677	1016.786	2.373	3483
SUTM	207.302	212.667	1.375	2028
SASS	348.567	343.420	1.457	2418

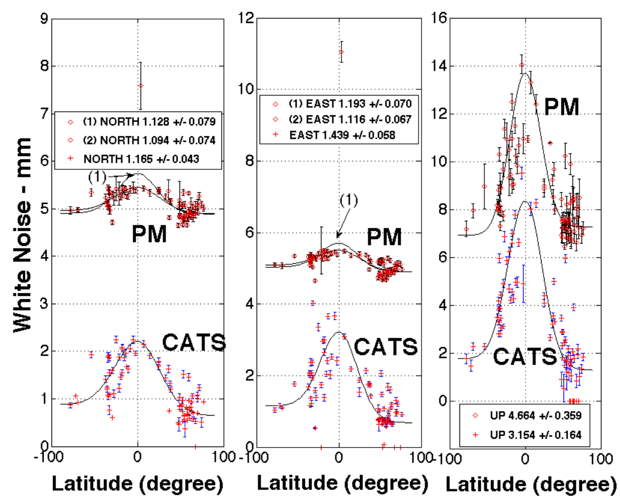

Fig. 4. GPS station latitude (degree) vs. estimated amplitude (and uncertainty) of the white noise with $CATS$ and the proposed algorithm (PM) on top of the hemisphere bias (solid line) with (see (1)) and whitout the large values. Note that the results for PM are offset by 4 mm.

Finally, the white noise is estimated for 78 GPS stations (e.g. Figure 2) for each coordinate (East, North and Up). The length of the time series is varied from 1000 to 3400 epochs (3 to 8 years).

Figures 4 displays the results. A hemisphere bias (solid line) is calculated by interpolating the equations given in [19] using the estimated white noise amplitude (with the proposed method or CATS software). Note CATS does not output the standard deviation values for 5 stations in East coordinate, 1 station in North coordinate, and 5 stations in Up coordinate. [20] explains this error either as a way of limiting CATS computation time, or when the values are higher than 1 sigma threshold. The overall results show that the proposed method is estimating a higher mean amplitude of the white noise in the Up coordinate ($P.M.$: 4.544 mm - CATS: 3.154 mm). As mentioned before, the uncertainty values of the white noise amplitude are also higher than the CATS results for the all three coordinates. However, the results for both algorithm show a large positive bias for the stations in the southern hemisphere indicating that the time series are noisier than in the northern hemisphere. These results were already noted in [19]. The results with the proposed method emphasize a much higher white noise amplitude for some stations very close to the equator on the East and North coordinates. The higher noise for the stations around the equator is a phenomenon which has recently received some attention and it may be explained by GPS signal propagation error (e.g. refraction) through the ionosphere [17]. The refractive index value is linked with the free electrons in the medium and it affects the velocity of the electro-magnetic waves causing the well known effects of first-order ionospheric phase advance and group delay. Recent studies, such as [13], have shown that this effect increases the white noise in GPS coordinate time series close to the equator as the total electron content is generally higher in this region.

The hemisphere bias is smaller in amplitude (0.25 mm - North to 0.6 mm - East) and in scatter with the proposed method than using CATS. Therefore, the effect of the total electron content in the white noise of GPS coordinates in the equatorial region may be less than previously shown (see [19]).

IV. Discussion and Conclusions

This work explains in detail an algorithm to quickly and efficiently estimate the white noise statistics from GPS coordinate time series. This gives an alternative to using the CATS software which is demanding in terms of computing time as it calculates the statistics for all components of the noise (white noise, flicker noise, random-walk).

Overall, the results from the proposed method agree with CATS. The process of this algorithm can be compared to a filter, and thus it quickly estimates the white noise statistics. The computation time is reduced by a factor of at least 100 compared to CATS. However, we show that the uncertainty values are sensitive to the criterion chosen (see Step 3 of the proposed method). Thus, the drawback of this algorithm is the reliability of the uncertainty of the estimated amplitude, which is generally slightly higher, or very small. In minor cases, the uncertainty values are high in comparison with the estimated amplitude of the white noise ($Un_w \sim 20\% Amp$). However, CATS sometime fails to calculate reliably the noises' amplitude (i.e due to short time series).

Due to its low complexity, this algorithm may be useful to check the level of the white noise in GPS coordinate time series, discarding noisy GPS time series prior to a GPS network analysis, improving the filtering of GPS time series, or checking the validity of estimated coseismic offsets.

The last step of this study shows the distribution of the estimated white noise statistics versus the latitude of 78 GPS stations. The results with the proposed method agree with the general conclusion that the GPS coordinate time series distributed close to the equator are noisier due to perhaps the well known effects of first-order ionospheric phase advance and group delay affecting the transmitted GPS signal. Finally this algorithm may open new research directions for new estimation techniques, based on the knowledge of the white noise statistics in GPS coordinate time series, in order to estimate the other noise components (i.e. flicker noise, random-walk).

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