THE CONNECTION BETWEEN ELASTODYNAMIC REPRESENTATION THEOREMS AND PROPAGATOR MATRICES

BY B. L. N. KENNETT

ABSTRACT

Two independent powerful methods in elastodynamics have been developed over the last few years: these are the use of representation theorems and propagator matrix techniques. The connection between these two approaches is derived in this paper. This is then used to show the equivalence of approximate techniques in elastic wave scattering based on representation theorems and a recent method based on matrix theory.

INTRODUCTION

In recent years two independent, powerful methods have been developed for handling problems in elastodynamics: these are the use of representation theorems and propagator matrix techniques. It is the purpose of the first part of this paper to illustrate the connection between these two approaches; the second part uses this connection to show the equivalence of the method introduced by Kennett (1972a, b) and previous work on elastodynamic scattering theory.

A representation theorem for the displacement in elastic wave problems, similar to Kirchhoff’s formula in scalar wave propagation (Baker and Copson, 1954), was first given by Kupradze (1950). Kupradze considered a homogeneous isotropic medium and harmonic time dependence; his work was subsequently extended by de Hoop (1958) to arbitrary time dependence and by Burridge and Knopoff (1964) to deal with an inhomogeneous anisotropic medium. On the other hand, a matrix method for considering elastic wave propagation in multilayered elastic media was introduced by Thomson (1950) and subsequently corrected and improved by Haskell (1953). Later Gilbert and Backus (1966) introduced the term “propagator” and provided a more formal mathematical background to the technique.

Both of these methods have been widely used in elastodynamics. Representation theorems have been used in exact solutions of diffraction problems (de Hoop, 1958) and by many authors in approximate methods (e.g., Skuridin, 1955; Knopoff and Hudson, 1964). The matrix methods have been used mainly to determine dispersion relations for surface waves and to solve source problems in multilayered elastic media (the layering may be either horizontal or homogeneous spherical shells).

Representation theorems have been used by Herrera (1964), Herrera and Mal (1965), and Hudson (1968) to obtain expressions for the scattering of elastic waves using a first-order perturbation technique. In Kennett (1972a, b) an alternative approach to the same class of problems has been derived based on the use of matrix methods. By using the relation between the representation theorem and the matrix methods derived in the first part of this paper, we are able to show that these two first-order approximate methods are equivalent as applied to these scattering problems.

REPRESENTATION THEOREMS

We consider a region \( V \), bounded by the surface \( S \), containing an anisotropic inhomogeneous medium with elastic constants \( c_{ijpq}(x) \). The components of the Green’s tensor
\( G_m(x, t, \xi, t') \) are defined as the displacement in the \( m \)th direction at \((x, t)\) due to an instantaneous point force of unit impulse in the \( i \)th direction at \((\xi, t')\), and are functions of the spatial and temporal separations \( x - \xi, t - t' \). We define the function \( \Theta \) by

\[
\Theta(x) = \begin{cases} 1 & x \in V, \\ 0 & x \not\in V, \end{cases}
\]

and we write \( n_j \) for the components of the outward unit normal to \( S \) at any point. The elastodynamic representation theorem for the \( n \)th component of displacement \( u_n(x, t) \) (referred to cartesian axes) in the presence of a body force distribution \( f(x, t) \) is given by (Burridge and Knopoff, 1964)

\[
\Theta(x) u_n(x, t) = \int_{-\infty}^{\infty} dt' \int_{\xi} \mathcal{G}_m(x, t, \xi, t') f_\xi(t') \, d^3\xi
\]

\[
+ \int_{-\infty}^{\infty} dt' \int_{\xi} n_j \mathcal{G}_m(x, t, \xi, t') c_{j\xi q}(\xi) \frac{\partial u_q}{\partial t}(\xi, t')
\]

\[
- u_i(\xi, t') c_{j\xi q}(\xi) \frac{\partial \mathcal{G}_m}{\partial \xi_q}(x, t, \xi, t') d\Sigma,
\]

(2)

where we have used the convention of summation over repeated suffices. We now take a Fourier transform with respect to time and write

\[
\hat{u}(x, \omega) = \int_{-\infty}^{\infty} e^{i\omega t} u(x, t) \, dt,
\]

(3)

and use the result that the Fourier transform of a convolution is the product of the Fourier transforms

\[
\mathcal{F}[\int_{-\infty}^{\infty} dt' f(t')g(t')] = \hat{f}(\omega) \hat{g}(\omega).
\]

We thus obtain

\[
\Theta(x) \hat{u}_n(x, \omega) = \int_{\xi} \hat{\mathcal{G}}_m(x - \xi, \omega) \hat{f}_\xi(\xi, \omega) \, d^3\xi
\]

\[
+ \int_{\xi} n_j \left( c_{j\xi q}(\xi) \left( \hat{\mathcal{G}}_m(x - \xi, \omega) \frac{\partial \hat{u}_q}{\partial \xi}(\xi, \omega)
\right.ight.
\]

\[
- \hat{u}_i(\xi, \omega) \frac{\partial \hat{\mathcal{G}}_m}{\partial \xi_q}(x - \xi, \omega)) \, d\Sigma.
\]

(5)

From the stress-strain relations for the elastic medium

\[
\hat{\tau}_{im}(x, \omega) = c_{imn}(x) \frac{\partial \hat{u}_n}{\partial x_m}(x, \omega)
\]

(6)

we may set up a similar but slightly more complicated representation theorem for the stresses

\[
\Theta(x) \hat{\tau}_{im}(x, \omega) = \int_{\xi} c_{imn}(x) \frac{\partial \hat{\mathcal{G}}_p}{\partial x_m}(x - \xi, \omega) \hat{f}_\xi(\xi, \omega) \, d^3\xi
\]

\[
+ \int_{\xi} n_j c_{imn}(\xi) \left( \frac{\partial \hat{\mathcal{G}}_p}{\partial x_m}(x - \xi, \omega) \frac{\partial \hat{u}_n}{\partial \xi_q}(\xi, \omega)
\right.
\]

\[
- \hat{u}_i(\xi, \omega) \frac{\partial^2 \hat{\mathcal{G}}_p}{\partial x_m \partial \xi_q}(x - \xi, \omega)) \, d\Sigma.
\]

(7)

### Propagator Matrices

For an isotropic medium or a transversely isotropic medium (with the \( x_3 \) axis as the axis of symmetry), whose properties depend only on the \( x_3 \)-coordinate, the equations of motion and the stress-strain relations can be combined to give a differential equation

\[
\partial_3 \mathbf{B}(k_1, k_2, x_3, \omega)/\partial x_3 = \mathbf{A}(k_1, k_2, x_3, \omega) \mathbf{B}(k_1, k_2, x_3, \omega)
\]

(8)

in terms of the stress-displacement vector \( \mathbf{B} \)

\[
\mathbf{B}(k_1, k_2, x_3, \omega) = \text{col} \left[ \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{\tau}_{13}, \bar{\tau}_{23}, \bar{\tau}_{33} \right]
\]

(9)
where the bar denotes Fourier transformation with respect to the coordinates \( x_1, x_2 \) and time, i.e.,

\[
\bar{u}(k_1, k_2, x_3, \omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ e^{i(\omega t - k_1 x_1 - k_2 x_2)} u(x, t). \tag{10}
\]
The matrix \( \bar{A}(k_1, k_2, x_3, \omega) \) introduced in (8) depends on the elastic properties at the level \( x_3 \), and for example for an isotropic medium with Lamé parameters \( \lambda, \mu \) and density \( \rho \), \( \bar{A} \) has the form

\[
\begin{bmatrix}
0 & 0 & -ik_1 & \mu^{-1} & 0 & 0 \\
0 & 0 & -ik_2 & 0 & \mu^{-1} & 0 \\
-ik_1 \lambda \chi^{-1} & -ik_2 \lambda \chi^{-1} & 0 & 0 & 0 & \chi^{-1} \\
vk_1^2 + \mu k_2^2 - \rho \omega^2 & \sigma k_1 k_2 & 0 & 0 & 0 & -ik_1 \lambda \chi^{-1} \\
\sigma k_1 k_2 & vk_2^2 + \mu k_1^2 + \rho \omega^2 & 0 & 0 & 0 & -ik_2 \lambda \chi^{-1} \\
0 & 0 & -\rho \omega^2 & -ik_1 & -ik_2 & 0
\end{bmatrix}
\tag{11}
\]
where

\[
\chi = \lambda + 2\mu, \\
v = 4\mu (\lambda + \mu) \chi^{-1}, \quad \sigma = \mu (3\lambda + 2\mu) \chi^{-1}.
\]

For a two-dimensional problem, i.e., one in which the elastic parameters and stress-displacement vector are independent of the coordinate \( x_2 \), the Fourier transformation over \( x_2 \) is redundant. The resulting matrix \( \bar{A} \), now a function of \( k_1 \) alone, may be obtained by setting \( k_2 = 0 \) in (11). The equations now separate into two groups:

(a) corresponding to \( P-SV \)-wave propagation with stress-displacement vector

\[
\bar{B}_P(k_1, x_3, \omega) = \text{col} [\bar{u}_1, \bar{u}_3, \bar{e}_{13}, \bar{e}_{33}] \tag{12}
\]

(b) corresponding to \( SH \) waves with

\[
\bar{B}_{SH}(k_1, x_3, \omega) = \text{col} [\bar{u}_2, \bar{e}_{23}] \tag{13}
\]

and both these vectors satisfy equations of the type (8) (Gilbert and Backus, 1966).

We therefore consider the system of differential equations

\[
\frac{\partial \bar{B}(z)}{\partial z} = \bar{A}(z) \bar{B}(z). \tag{14}
\]

The propagator matrix for equation (14) is defined as the matrix solution of

\[
\frac{\partial P(z, z_0)}{\partial z} = \bar{A}(z) P(z, z_0) \tag{15}
\]

such that

\[
P(z_0, z_0) = 1 \tag{16}
\]

where 1 is the unit matrix (Gilbert and Backus, 1966). The solution of (14) with initial value \( \bar{B}(z_0) \) is then

\[
\bar{B}(z) = P(z, z_0) \bar{B}(z_0). \tag{17}
\]

Such a propagator matrix is unique; for suppose that \( P(z) \) is one solution of (15) with \( P(z_0) = P_0 \) [this will be nonsingular for any value of the argument \( z \) (Gantmacher, 1959, p. 136)], and \( Q(z) \) is another solution satisfying the same conditions then

\[
\frac{\partial Q}{\partial z} = \frac{\partial [P^{-1} Q]}{\partial z} = \frac{\partial P}{\partial z} (P^{-1} Q) + P \frac{\partial (P^{-1} Q)}{\partial z} = \bar{A} Q + P \frac{\partial (P^{-1} Q)}{\partial z}.
\]
Thus \( \partial (P^{-1} Q) / \partial z = 0 \), as \( P \) is nonsingular; and \( P^{-1}(z) Q(z) \) is, therefore, a constant, independent of \( z \), i.e.,

\[
P^{-1}(z) Q(z) = P^{-1}(z_0) Q(z_0) = 1
\]

and, therefore, \( P(z) \equiv Q(z) \) so that the solution is unique.

The propagator matrix satisfies

\[
P(z, z_0) = P(z, \zeta) P(\zeta, z_0)
\]

and, therefore, bears a simple relation to its inverse

\[
P^{-1}(z, z_0) = P(z_0, z).
\]

The inhomogeneous equation of the form

\[
\partial B(z) / \partial z = \Lambda(z) B(z) + \beta(z)
\]

has the solution

\[
B(z) = \int_{x_0}^{x} P(z, y) \beta(y) dy + P(z, z_0) B(z_0)
\]

indicating the nature of the propagator as a Green's function for (14).

**The Connection Formula**

For convenience we shall now consider only two-dimensional problems (in which all stresses and displacements are independent of \( x_2 \)), although the extension to three dimensions may readily be performed. We consider an elastic medium whose properties depend only on the coordinate \( x_3 \) and apply the representation theorems (5) and (7) to the region \( V \) bounded by the planes \( x_3 = z_0, x_3 = z_1 \), and \( |x_1| = R \).

![Fig. 1. Geometry of the region.](image)

We will assume \( \hat{G}_{mn} \) to be the Green's tensor for an unbounded medium. As \( R \to \infty \) the contributions to the surface integrals from the portions of the surface \( S \) parallel to the \( x_3 \) axis, i.e., the parts \( |x_1| = R, z_0 < x_3 < z_1 \), may be neglected since \( \hat{G}_{mn}(x - \xi, \omega) \) behaves like \( R^{-1/2} \) for \( x \) on these portions.

We define the function \( \Theta(x_3) \) as

\[
\Theta(x_3) = \begin{cases} 1 & x_3 \in (z_0, z_1), \\ 0 & x_3 \notin (z_0, z_1), \end{cases}
\]

and using the equation (5) we have the following representation for the displacement corresponding to \( P-SV \)-wave propagation

\[
\Theta(x_3) \hat{u}_m(x_1, x_3, \omega) = \int_{x_0}^{x} \int_{-\infty}^{\infty} \hat{G}_{mn}(x_1 - \xi_1, x_3 - \xi_3, \omega) \tilde{f}_n(\xi_1, \xi_3) \, d\xi_1 \, d\xi_3 \\
+ J_m(x_1, x_3, z_1) - J_m(x_1, x_3, z_0)
\]

(22)
ELASTODYNAMIC REPRESENTATION THEOREMS AND PROPAGATOR MATRICES

The suffices here take the values 1, 3. The expressions \( J_m(x_1, x_3, \xi_3) \) introduced in (22) are line integrals arising from the surface integral in (5)

\[
J_m(x_1, x_3, \xi_3) = \int_{-\infty}^{\infty} \frac{d^2 q}{(2\pi)^2} \left( \hat{\theta}_{mj}(x_1 - \xi_1, x_3 - \xi_3, \omega) \right) \hat{u}_{m}(\xi_1, \xi_3) \partial \hat{\theta}_{mj} / \partial q_1 \partial \hat{\theta}_{mj} / \partial q_3
\]

where we have written \( \hat{\theta}_{mj} = \hat{\theta}_{mj}(x_1 - \xi_1, x_3 - \xi_3, \omega) \). There is a similar representation for the stresses. To cast these results into a more convenient form, we now Fourier transform these representation theorems with respect to \( x_1 \). Hence, writing

\[
\hat{u}(k, x_3, \omega) = \int_{-\infty}^{\infty} e^{-ikx_1} \hat{u}(x_1, x_3, \omega) \, dx_1,
\]

we obtain the representation for the displacement as

\[
\Theta(x_3) \hat{u}_m(k, x_3, \omega) = \int \frac{dx_1}{2\pi} \left[ \hat{\theta}_{mj}(k, x_3 - \xi_3, \omega) \hat{f}_j(k, \xi_3, \omega) \, d\xi_3 + \hat{J}_m(k, x_3, z_1) - \hat{J}_m(k, x_3, z_0) \right]
\]

and for the stresses

\[
\Theta(x_3) \hat{\tau}_{in}(k, x_3, \omega) = \int \frac{dx_1}{2\pi} \left[ \hat{\theta}_{mj}(k, x_3 - \xi_3, \omega) \hat{f}_j(k, \xi_3, \omega) \, d\xi_3 + \hat{J}_m(k, x_3, z_1) - \hat{J}_m(k, x_3, z_0) \right]
\]

where \( \hat{\theta}_{pj} = \hat{\theta}_{pj}(x_3 - \xi_3, \omega) \). The transformed line integrals appearing in the expressions (25) and (26) have the form

\[
\hat{J}_m(k, x_3, \xi_3) = c_{j3r3} \left[ \hat{\theta}_{mj} \partial \hat{u}_r / \partial \xi_3(k, \xi_3) - \hat{u}_j(k, \xi_3) \partial \hat{\theta}_{mr} / \partial \xi_3 \right] + c_{j3r1} \left[ \hat{\theta}_{mj} \partial \hat{u}_r / \partial \xi_3(k, \xi_3) - \hat{u}_j(k, \xi_3) \partial \hat{\theta}_{mr} / \partial \xi_3 \right]
\]

If we now consider a region free from body force and take \( x_3 \) to lie outside \((z_0, z_1)\), from (25) and (26) we require

\[
\hat{J}_m(k, x_3, z_1) = \hat{J}_m(k, x_3, z_0), \quad m = 1, 3
\]

for all \( x_3 \notin (z_0, z_1) \). In the case of an isotropic medium or a transversely isotropic medium with the \( x_3 \) axis as the axis of symmetry, \( \hat{J}_m \) can be expressed in terms of \( \hat{u}_1, \hat{u}_3, \hat{\tau}_{11}, \hat{\tau}_{13} \) alone and the four equations (28) can be written in the form

\[
H(k, x_3, z_1) \hat{B}_p(k, z_1) = H(k, x_3, z_0) \hat{B}_p(k, z_0), \quad x_3 \notin (z_0, z_1)
\]

in terms of the stress-displacement vector \( \hat{B}_p \) introduced in equation (12). For these media the matrix \( H(k, x_3, z) \) has the form

\[
H(k, x_3, z) = \left[ \begin{array}{ccc} H_{11} & H_{13} & \bar{\theta}_{11} \\ H_{31} & H_{33} & \bar{\theta}_{31} \\ \partial H_{11} / \partial x_3 & \partial H_{13} / \partial x_3 & \partial \bar{\theta}_{11} / \partial x_3 \\ \partial H_{31} / \partial x_3 & \partial H_{33} / \partial x_3 & \partial \bar{\theta}_{31} / \partial x_3 \end{array} \right]
\]

where

\[
H_{11} = -c_{13p3} \partial \bar{\theta}_{1p} / \partial z - ik c_{13p1} \bar{\theta}_{1p} = -\tau_{13}(\bar{\theta}_1),
\]

and similarly

\[
H_{13} = -\tau_{33}(\bar{\theta}_1), \quad H_{31} = -\tau_{33}(\bar{\theta}_3), \quad H_{33} = -\tau_{33}(\bar{\theta}_3)
\]

We have written here

\[
\bar{\theta}_{1j} = \bar{\theta}_{1j}(k, x_3 - z, \omega),
\]

and also \( \bar{\theta}_j \) for the displacement vector at the level \( x_3 \) because of a unit force in the \( j \)th direction at the level \( z \). With the assumption that \( \bar{\theta}_{1j} \) is the Green’s tensor for an un-
bounded medium, the matrix \( \mathbf{H} \) is nonsingular, so that we may write
\[
\mathbf{B}_p(k, z_{1}) = \mathbf{H}^{-1}(k, x_{3}, z_{1}) \mathbf{H}(k, x_{3}, z_{0}) \mathbf{B}_p(k, z_{0}), \quad x_{3} \not\in (z_{0}, z_{1})
\] (34)
comparing this expression with (17), we see that it is suggestive of the form for a propagator matrix
\[
\mathbf{B}_p(k, z_{1}) = \mathbf{P}(k, z_{1}, z_{0}) \mathbf{B}_p(k, z_{0}).
\] (35)

We now show that \( \mathbf{H}^{-1}(k, x_{3}, z_{1}) \mathbf{H}(k, x_{3}, z_{0}) \) is indeed the propagator matrix for the medium. Defining
\[
\mathbf{Q}(k, z_{1}, z_{0}) = \mathbf{H}^{-1}(k, x_{3}, z_{1}) \mathbf{H}(k, x_{3}, z_{0}), \quad x_{3} \not\in (z_{0}, z_{1})
\] (36)
we find
\[
\mathbf{Q}^{-1}(k, z_{1}, z_{0}) = \left[ \mathbf{H}^{-1}(k, x_{3}, z_{1}) \mathbf{H}(k, x_{3}, z_{0}) \right]^{-1}
\] (37)
\[
= \mathbf{H}^{-1}(k, x_{3}, z_{0}) \mathbf{H}(k, x_{3}, z_{1})
\]
and also
\[
\mathbf{Q}(k, z_{0}, z_{0}) = \mathbf{H}^{-1}(k, x_{3}, z_{0}) \mathbf{H}(k, x_{3}, z_{0}) = \mathbf{I}. \] (38)

Consider now differentiating equation (34), we obtain
\[
\frac{\partial \mathbf{B}_p(k, z)}{\partial z} = \frac{\partial}{\partial z} \left[ \mathbf{Q}(k, z, z_{0}) \mathbf{B}_p(k, z_{0}) \right]
\]
and since \( \frac{\partial \mathbf{B}_p(k, z)}{\partial z} = \overline{\mathbf{A}}_p(k, z) \mathbf{B}_p(k, z) \) for an isotropic or transversely isotropic medium,
\[
\frac{\partial}{\partial z} \left[ \mathbf{Q}(k, z, z_{0}) \right] \mathbf{B}_p(k, z_{0}) = \overline{\mathbf{A}}_p(k, z) \mathbf{Q}(k, z, z_{0}) \mathbf{B}_p(k, z_{0}).
\] (39)
This last equation must hold for arbitrary initial values \( \mathbf{B}_p(k, z_{0}) \) and initial positions \( z_{0} \), so that we have finally
\[
\frac{\partial \mathbf{Q}(k, z, z_{0})}{\partial z} = \overline{\mathbf{A}}_p(k, z) \mathbf{Q}(k, z, z_{0}),
\] (40)
and from (38) \( \mathbf{Q}(k, z_{0}, z_{0}) = \mathbf{I} \). Thus, from the uniqueness of the propagator matrix, we may identify the propagator corresponding to \( \overline{\mathbf{A}}_p \) with \( \mathbf{Q}(k, z, z_{0}) \).

We have, therefore, established the required connection between the representation theorem and the propagator matrix for an isotropic or transversely isotropic medium, through
\[
\mathbf{P}(k, z_{1}, z_{0}) = \mathbf{H}^{-1}(k, x_{3}, z_{1}) \mathbf{H}(k, x_{3}, z_{0}), \quad x_{3} \not\in (z_{0}, z_{1}).
\] (41)
A similar construction may be given for \( \mathbf{S} \mathbf{H} \) waves, (41) again holds with \( \mathbf{H} \) replaced by
\[
\mathbf{H}_{\mathbf{S} \mathbf{H}} = \begin{pmatrix}
    h_{23} & \frac{\partial g_{23}}{\partial x_{3}} \\
    \frac{\partial h_{23}}{\partial x_{3}} & \frac{\partial g_{23}}{\partial x_{3}}
\end{pmatrix},
\] (42)
where
\[
h_{23} = -\tau_{23}(\mathbf{g}_{2}).
\] Although the propagator is itself unique, the matrices in (41) are not unique for as pointed out by Gilbert and Backus [1966, equation (2.4)]
\[
\mathbf{P}(k, z_{1}, z_{0}) = \mathbf{M}(z_{1}) \mathbf{M}^{-1}(z_{0}),
\]
provided that \( \mathbf{M}(z) \) satisfies
\[
\frac{\partial \mathbf{M}(z)}{\partial z} = \overline{\mathbf{A}}(k, z) \mathbf{M}(z).
\]
Different matrices \( \mathbf{H}(k, x_{3}, z_{1}) \) may be generated by choosing for \( \mathbf{g}_{x_{j}} \) a Green’s tensor satisfying certain boundary conditions, provided that these are applied outside the region under consideration.

If we now reintroduce body forces into the region \( (z_{0}, z_{1}) \) equation (29) becomes
ELASTODYNAMIC REPRESENTATION THEOREMS AND PROPAGATOR MATRICES

\[ \mathbf{H}(k, x_3, z_1) \mathbf{B}_p(k, z_1) = \mathbf{H}(k, x_3, z_0) \mathbf{B}(k, z_0) + \int_{z_0}^{z_1} \mathbf{H}(k, x_3, \zeta) \mathbf{F}_p(k, \zeta) \, d\zeta, \quad x_3 \neq (z_0, z_1), \]

(43)

where the vector \( \mathbf{F}_p \) depends on the body force

\[ \mathbf{F}_p(k, z) = \begin{bmatrix} 0 \\ 0 \\ -f_1(k, z, \omega) \\ -f_3(k, z, \omega) \end{bmatrix}. \]

(44)

Thus using equation (41) we obtain the following expression in terms of the propagator

\[ \mathbf{B}_p(k, z_1) = P(k, x_3, z_1) \mathbf{B}_p(k, z_0) + \int_{z_0}^{z_1} P(k, z, \zeta) \mathbf{F}_p(k, \zeta) \, d\zeta. \]

(45)

We may note that equation (45) is in the same form as equation (20) and is thus simply the result of solving the set of equations

\[ \partial \mathbf{B}_p(k, z)/\partial z = \mathbf{A}_p(k, z) \mathbf{B}_p(k, z) + \mathbf{F}_p(k, z) \]

(46)

which is just a rearrangement of the equations of motion in the presence of a body force.

The body forces may also be regarded as equivalent to stress discontinuities across planes perpendicular to the \( x_3 \) axis (Burridge and Knopoff, 1964); the introduction of such discontinuities is a common method of introducing sources into matrix techniques. This correspondence is discussed at greater length by Hudson (1969). If we have a body force \((f_1, f_3)\) at the point \((0, \zeta)\), this is equivalent to a discontinuity in the vector \( \mathbf{B}_p \) across the plane \( x_3 = \zeta \)

\[ \mathbf{B}_p(k, \zeta+) - \mathbf{B}_p(k, \zeta-) = \begin{bmatrix} 0 \\ 0 \\ -f_1 \\ -f_3 \end{bmatrix} \]

(47)

and we see that (45) may also be regarded as an integration over a distribution of discontinuities, the effect of each discontinuity propagating to the level \( x_3 = z \). More complicated sources may be represented in terms of a more elaborate source vector \( \mathbf{F}_p \).

COMPARISON OF TWO SCATTERING FORMALISMS

Most previous work on the general theory of elastic wave scattering has used first-order perturbation theory coupled with the use of representation theorems to obtain expressions for the scattered displacement caused by a given incident wave. For example, following Hudson (1968) we consider a distribution of elastic parameters

\[ \rho_{ijpq} = \rho_{ijpq}^0 + \rho_{ijpq}', \quad \rho = \rho^0 + \rho', \]

(48)

where

\[ \frac{\rho_{ijpq}'}{\rho^0} \ll 1, \quad \frac{\rho'}{\rho^0} \ll 1, \]

(49)

under the assumption that \( \rho_{ijpq} \) and \( \rho' \) are piecewise continuous and differentiable, and nonzero only in a finite region \( R \). For an isotropic medium

\[ c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}). \]

(50)

A harmonic wave incident on \( R \) may be described by the displacement \( \mathbf{u}^0 e^{-i\omega t} \) and the resulting displacement by

\[ \mathbf{u} e^{-i\omega t} = (\mathbf{u}^0 + \mathbf{u}') e^{-i\omega t}, \]

(51)

we assume that \( |\mathbf{u}'| \ll |\mathbf{u}^0| \) everywhere.

For an unbounded medium we may quote the result of Hudson, that taken to a first-order approximation

\[ u'_m(x) = \int_R \{ \rho' \omega^2 u_i^0(\xi) \hat{G}_{mi}(x, \xi) - c_{ijpq} \partial u_p^0/\partial \xi_q (\xi) \partial \hat{G}_{mi}/\partial \xi_j (x, \xi) \} \, d^3\xi \]

(52)
where $\mathbf{G}_m(x, \xi)$ is the Green's tensor for an unbounded, not necessarily homogeneous, medium with properties $\epsilon^{i}^{j}_{pq}$, $\rho^{0}$. The scattering effect of the region $R$ has here been introduced as a volume distribution of source terms depending on the incident field. We may now apply the representation theorem to a bounded region $V$ and obtain

$$\Theta(x)u'_m(x) = \int_{R'} \{ \rho' \omega^2 u_i^{0}(\xi) \mathbf{G}_m(x, \xi) - c_{ijpq}^{0} \partial u_p^{0}/\partial \xi_q(x, \xi) \partial \mathbf{G}_m/\partial \xi_j(x, \xi) \} d^3 \xi + \int_{S} n_j c_{ijpq}^{0} \{ \partial \mathbf{G}_m(x, \xi) / \partial \xi_q(x, \xi) - u_i^{0}(\xi) \partial \mathbf{G}_m/\partial \xi_i(x, \xi) \} dS$$

(53)

where $R'$ is that portion of $R$ contained in $V$. However, because $\rho'$, $c_{ijpq}^{0}$ are zero outside $R'$, the volume integration may be extended to the whole of $V$.

We now specialize to a two-dimensional problem and consider isotropic media whose elastic properties depend only on the $x_a$-coordinate. We take $V$ to be the region between the planes $x_3 = z_0$, $x_3 = z_1$ and apply a Fourier transform with respect to $x_3$ to (53) and the corresponding representation in terms of the stresses, using the result that the Fourier transform of a product is the convolution of the Fourier transforms

$$F(u(x), \nu(x)) = 1/2\pi \tilde{u} * \tilde{v}(k)$$

(54)

we find that the displacement representation has the form

$$\Theta(x_3) \tilde{u}_m(k, x_3, \omega) = (1/2\pi) \int_{z_0}^{z_1} d\xi_3 \int_{-\infty}^{\infty} d\eta \{ \tilde{\rho}' \tilde{u}_j^{0}(\eta) \tilde{\mathbf{G}}_{mj}(k) + \tilde{\eta} \tilde{c}_{j1pq}^{0}(\eta) \tilde{\mathbf{G}}_{mj}(k)$$

$$+ \tilde{\epsilon}_{j1pq}^{0}(\eta) \tilde{\mathbf{G}}_{mj}(k) + \eta \tilde{c}_{j3pq}^{0}(\eta) \tilde{\mathbf{G}}_{mj}(k) + \eta \tilde{\epsilon}_{j3pq}^{0}(\eta) \tilde{\mathbf{G}}_{mj}(k)$$

$$+ \tilde{J}_m(k, x_3, z_1) - \tilde{J}_m(k, x_3, z_0)$$

(55)

where we have written

$$\tilde{\rho}' = \tilde{\rho}'(k - \eta, \xi_3), \quad \tilde{c}_{j1pq}^{0} = \tilde{c}_{j1pq}^{0}(k - \eta, \xi_3)$$

and the expressions $\tilde{J}_m(k, x_3, \xi_3)$ were defined in equation (27). A similar but more complicated expression may be found for the stress representation. Using these expressions we find that if we take $x_3$ to lie outside $(z_0, z_1)$ as in the previous section, after some manipulation these representations may be written as

$$\mathbf{H}(k, x_3, z_1) \mathbf{B}'_p(k, z_1) = \mathbf{H}(k, x_3, z_0) \mathbf{B}'_p(k, z_0)$$

$$+ \frac{1}{2\pi} \int_{z_0}^{z_1} \mathbf{H}(k, x_3, \xi_3) \int_{-\infty}^{\infty} \bar{\mathbf{S}}_p(k, \eta, \xi_3) d\eta d\xi_3, \quad x_3 \not\in (z_0, z_1),$$

(56)

where the source vector $\bar{\mathbf{S}}_p$ depends on the incident wave and the spectral decomposition of the elastic contrasts between inclusion and matrix $\lambda'(k)$, $\bar{\nu}'(k)$, $\tilde{\rho}'(k)$. Using equation (41) we see that for $P-SV$ waves (56) can be rewritten in the form

$$\mathbf{B}_p(k, z_1) = \mathbf{P}(k, z_1, z_0) \mathbf{B}_p(k, z_0) + \frac{1}{2\pi} \int_{z_0}^{z_1} \mathbf{P}(k, z_1, \xi_3) \int_{-\infty}^{\infty} \bar{\mathbf{S}}_p(\eta, \xi_3) d\eta d\xi_3.$$  

(57)

The source vector $\bar{\mathbf{S}}_p$ has the form

$$\bar{\mathbf{S}}_p(k, \eta, \xi_3) = \text{col} [s_1, s_2, s_3, s_4]$$

(58)
where

\[
\begin{align*}
I_{s1}(k, \eta, \xi_3) &= -\frac{1}{\mu_0^2} \tilde{\mu}' \tilde{\tau}_{13}^0(\eta, \xi_3) \\
I_{s2}(k, \eta, \xi_3) &= -i\eta \frac{\tilde{\lambda}'}{\lambda_0 + 2\mu_0} \tilde{u}_1^0(\eta, \xi_3) - \frac{\tilde{\lambda}' + 2\tilde{\mu}'}{\lambda_0 + 2\mu_0} \tilde{\tau}_{13}^0(\eta, \xi_3) \\
I_{s3}(k, \eta, \xi_3) &= \{\eta k(\tilde{\lambda}' + 2\tilde{\mu}') - \tilde{\rho}' \omega^2\} \tilde{u}_1^0(\eta, \xi_3) - i\kappa \lambda_0 \tilde{u}_2^0 + i\kappa \tilde{\xi}_{13}^0(\eta, \xi_3) \\
I_{s4}(k, \eta, \xi_3) &= -\tilde{\rho}' \omega^2 \tilde{u}_3^0(\eta, \xi_3)
\end{align*}
\]

and we have written

\[
\lambda' = \tilde{\lambda}'(k - \eta), \quad \mu' = \tilde{\mu}'(k - \eta), \quad \rho' = \tilde{\rho}'(k - \eta).
\]

The unperturbed parameters \(\lambda_0, \mu_0, \rho_0\), i.e., the elastic parameters of the matrix material are functions of \(\xi_3\) alone. The expressions (59) may be simplified slightly by using the relation

\[
\frac{\partial \tilde{u}_3^0}{\partial \xi_3}(\eta, \xi_3) = -i\eta \lambda_0 \tilde{u}_1^0(\eta, \xi_3) + \frac{1}{\lambda_0 + 2\mu_0} \tilde{\tau}_{13}^0(\eta, \xi_3)
\]

which gives

\[
\begin{align*}
I_{s2}(k, \eta, \xi_3) &= -2i\eta \left(\frac{\tilde{\lambda}' \mu_0 - \tilde{\mu}' \lambda_0}{(\lambda_0 + 2\mu_0)^2}\right) \tilde{u}_1^0(\eta, \xi_3) - \frac{\tilde{\lambda}' + 2\tilde{\mu}'}{(\lambda_0 + 2\mu_0)^2} \tilde{\tau}_{13}^0(\eta, \xi_3) \\
I_{s3}(k, \eta, \xi_3) &= \eta k \left[\left(\tilde{\lambda}' + 2\tilde{\mu}'\right) - \lambda_0 \left(\frac{\tilde{\lambda}' \mu_0 - \tilde{\mu}' \lambda_0 + 4\tilde{\lambda}' \mu_0}{(\lambda_0 + 2\mu_0)^2}\right)\right] \tilde{u}_1^0(\eta, \xi_3) \\
&\quad - \tilde{\rho}' \omega^2 \tilde{u}_1^0(\eta, \xi_3) - 2i\kappa \left(\frac{\tilde{\lambda}' \mu_0 - \tilde{\mu}' \lambda_0}{(\lambda_0 + 2\mu_0)^2}\right) \tilde{\tau}_{13}^0(\eta, \xi_3).
\end{align*}
\]

An alternative approach to this same elastic wave problem has been given in Kennett (1972a). That method was based on matrix techniques. For piecewise continuous and differentiable perturbations of the elastic parameters, the first-order scattered stress-displacement field may be found from

\[
\widehat{\mathbf{B}}_p'(k, z_1) = \mathbf{P}(k, z_1, z_0) \widehat{\mathbf{B}}_p'(k, z_0) + \frac{1}{2\pi} \int_{z_0}^{z_1} d\xi_3 \mathbf{P}(k, z_1, \xi_3) \int_{-\infty}^{\infty} C(k, \eta, \xi_3) \widehat{\mathbf{B}}_p^0(\eta, \xi_3) \, d\eta,
\]

which we see to be of the same form as equation (57). The matrix \(C(k, \eta, \xi_3)\) depends on the Fourier components of the departures of the elastic moduli and density from their unperturbed values and is given by

\[
C(k, \eta, \xi_3) =
\begin{bmatrix}
0 & 0 & \bar{a}_1(k - \eta, \xi_3) & 0 \\
\eta \bar{a}_3(k - \eta, \xi_3) & 0 & 0 & \bar{a}_2(k - \eta, \xi_3) \\
\eta k \bar{a}_4(k - \eta, \xi_3) - \omega^2 \bar{a}_5(k - \eta, \xi_3) & 0 & 0 & ik \bar{a}_3(k - \eta, \xi_3) \\
0 & -\omega^2 \bar{a}_5(k - \eta, \xi_3) & 0 & 0
\end{bmatrix}
\]
where to first order
\[
\begin{align*}
    a_1 &= \mu^{-1} - \mu_0^{-1} = -\mu' / \mu_0^2 \\
    a_2 &= (\lambda + 2\mu)^{-1} - (\lambda_0 + 2\mu_0)^{-1} = -(\lambda' + 2\mu') / (\lambda_0 + 2\mu_0)^2 \\
    a_3 &= \frac{\lambda}{\lambda + 2\mu} + \frac{\lambda_0}{\lambda_0 + 2\mu_0} = -2(\mu_0 - \mu') / (\lambda_0 + 2\mu_0)^2 \\
    a_4 &= \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} - \frac{4\mu_0(\lambda_0 + \mu_0)}{\lambda_0 + 2\mu_0} = \lambda' + 2\mu' - \lambda_0 \left[ \frac{\lambda' \lambda_0 - 2\mu' \lambda_0 + 4\mu_0 \lambda'}{(\lambda_0 + 2\mu_0)^2} \right] \\
    a_5 &= \rho - \rho_0 = \rho'.
\end{align*}
\]

On forming the matrix product \( C(k, \eta, \xi_3) \bar{B}_p^{(0)}(\eta, \xi_3) \), where
\[
\bar{B}_p^{(0)}(\eta, \xi_3) = \text{col} [u_1^{(0)}(\eta, \xi_3), u_3^{(0)}(\eta, \xi_3), \tau_1^{(0)}(\eta, \xi_3), \tau_3^{(0)}(\eta, \xi_3)],
\]
we see that
\[
\bar{S}_p(k, \eta, \xi_3) = C(k, \eta, \xi_3) \bar{B}_p^{(0)}(\eta, \xi_3)
\]
so that the two approaches via representation theory or by propagator matrix techniques are entirely equivalent.

The differences in the form of the scattering vectors in the two cases arises from their derivations. In the representation theory method we extract the matrix \( H(k, x_3, \xi_3) \) from the original source term to leave \( \bar{S}_p(k, \eta, \xi_3) \); i.e., we here have the final source terms as multiplying factors for Green’s tensor coefficients. In the matrix method the stress-displacement is the main object of interest and so this is extracted as a separate factor.

Because the two methods we have discussed are equivalent, either may be used in tackling any particular problem. The representation theorem method is probably most convenient for deriving results on the general nature of elastic wave scattering, whereas the matrix method is particularly suited to determining the scattering from a specified inhomogeneity in a multilatered elastic medium.

**ACKNOWLEDGMENT**

I would like to thank Dr. E. R. Lapwood for advice and encouragement and the Chairman and Directors of the British Petroleum Company Limited for the award of a Research Studentship and permission to publish this paper.

**REFERENCES**


Department of Applied Mathematics
and Theoretical Physics
Silver Street
Cambridge, CB3 9EW, England

Manuscript received January 14, 1972