SCATTERING APPROXIMATIONS FOR THICK AND THIN SCATTERERS

BY B. L. N. KENNETT

ABSTRACT

Two methods for handling seismic scattering problems have been developed in the last few years: one applies to interfacial irregularities or obstacles on interfaces, the other to larger scatterers. This paper examines some properties of these two methods and the relation between them.

INTRODUCTION

Most of the previous work on the scattering of seismic waves has concentrated on the effects of interface irregularities or of obstacles on interfaces (e.g., Gilbert and Knopoff, 1960; Hudson, 1967). In those papers, the theory has been developed in terms of a perturbation expansion for small departures from a plane interface. An alternative approach has considered the effect of scatterers which are no longer required to be thin, embedded in a matrix material (e.g., Herrera and Mal, 1965; Hudson, 1968).

The present author has recently published convenient methods for examining both of these types of problems when considering scattering phenomena in multi-layered media (Kennett, 1972a, b). The purpose of this note is to clarify the relation between the approach based on thin scatterer approximations and that for scatterers of greater thickness.

COMPARISON OF SCATTERING APPROXIMATIONS

For convenience in exposition, we will consider only two-dimensional problems and assume a time-dependence of the form \( \exp(-i\omega t) \). Both of the methods under consideration are based on the assumption that the total stress and displacement field in the presence of a scatterer

\[
B(x, z) = B^o(x, z) + B'(x, z)
\]

(1)

where \( B^o \) is the field in its absence and \( B' \) represents a scattered contribution. For a P or SV wave, e.g.,

\[
B(x, z) = \col [u, v, w, \tau_{xz}, \tau_{zz}]
\]

where \((u, v, w)\) is the displacement vector and \(\tau_{ij}\) the stress tensor. In each method, the scattered contribution \( B' \) may be expressed as the resultant of source terms introduced in the region of the scatterer depending on \( B \). We will take a first-order approximate approach and assume that \( B^o \) is an adequate approximation to the total field in the calculation of the sources (i.e., \(|B'| \ll |B^o|\)); this requires either small perturbations in the elastic parameters or small scatterers. The differences in the methods arise in the nature of the source distributions introduced.

Consider a scattering region of finite horizontal extent bounded above by the plane \( z = z_1 \) and below by \( z = z_2 \) (Figure 1); then using the results of Kennett (1972a) the scattered wave fields on the two sides of the scatterer are connected by

\[
B'(k, z_2) = P(k, z_2, z_1) \bar{B}(k, z_1)
\]

\[
+ (1/2\pi) \int_{z_1}^{z_2} d\xi' \int_{-\infty}^{\infty} d\xi P(k, z_2, \xi') C(k, \xi, \xi') \bar{B}^o(\xi, \xi')
\]

(2)

1321
where the overbar denotes Fourier transformation with respect to $x$. This source problem has to be solved subject to boundary conditions appropriate to the configuration, and represents the introduction of a volume distribution of sources dependent on the incident field. In equation (2) $P(k, z_2, z_1)$ is the propagator matrix for the medium surrounding the scatterer and $C(k, \xi, \zeta)$ depends on the Fourier components of the departures of the elastic parameters from lateral homogeneity. The shape of the scatterer appears through the depth variation of $C(k, \xi, \zeta)$.

For a scatterer whose internal elastic properties depend only on the depth $z$, bounded by a surface of discontinuity in the elastic parameters, we are able to factorize the scattering source matrix into a part dependent on the shape and a part depending only on the contrasts in elastic moduli and density across the scatterer boundary. Thus we may write

$$C(k, \xi, z) = f(k - \xi, z)L(k, \xi, z)$$

(3)

where $f(x, z)$ is a rectangular function of unit height within the scatterer and zero outside, and $L$ depends only on the contrasts in elastic parameters.

For example, for $P$ or $SV$ waves incident from a medium with properties $\lambda, \mu, \rho$ onto an inclusion with constant properties $\lambda', \mu', \rho'$, $L$ has the form

$$L(k, \xi, z) =$$

$$\begin{bmatrix}
0 & 0 & 1/\mu' - 1/\mu & 0 \\
\xi[\lambda'/\lambda + 2\mu'] - \lambda/(\lambda + 2\mu) & 0 & 0 & [1/(\lambda' + 2\mu') - 1/(\lambda + 2\mu)] \\
\xi k[v' - v] - [\rho' - \rho] \omega^2 & 0 & 0 & ik[\lambda'/\lambda + 2\mu'] - \lambda/(\lambda + 2\mu) \\
0 & 0 & -[\rho' - \rho] \omega^2 & 0
\end{bmatrix}$$

(4)

where $v = 4\mu(\lambda + \mu)/(\lambda + 2\mu)$ and similarly for $v'$.

If we now perform a Fourier inversion on equation (2), with $C$ taking the form (3), and look at the spatial distribution of the source terms we are introducing to describe the scattering effect of such an obstacle, we find that we have a combination of sources concentrated on the perimeter of the scatterer and a volume distribution of sources through the region of the scatterer. The perimeter terms obtained here correspond to the surface integral in the formulation via representation theorems (Herrera and Mal, 1965).

For a "thin" scatterer, i.e., one in which $H = z_2 - z_1$ is small, we make the assumption that the scattered field is linear in $H$ and use an alternative approximation (Kennett, 1972b). We here define the shape as an explicit function of $z$ and for a scatterer with
upper and lower boundary surfaces \( z = g_1(x), z = g_2(x) \) straddling the plane \( z = z_0 \) (Figure 1) the scattered fields on the two sides of \( z = z_0 \) are related by

\[
\mathcal{B}'(k, z_0^+) = \mathcal{B}'(k, z_0^-) + (1/2\pi) \int_{-\infty}^{\infty} d\xi [\tilde{g}_2(k-\xi) - \tilde{g}_1(k-\xi)] L(k, \xi, z_0) \mathcal{B}'(\xi, z_0)
\]

providing the slopes of the boundary surfaces are small. For a given combination of scatterer and matrix properties \((\lambda, \mu, \rho \text{ and } \lambda', \mu', \rho' \text{ say})\) the \( L(k, \xi, z_0) \) appearing in equation (5) will be identical to that in equation (3).

In the thin scatterer approximation (equation 5) the scattered wave is generated by a weighted distribution of sources along a single line. These sources depend explicitly on the shape of the obstacle, on the parameter contrasts, and the incident field at the line. The approximation developed for thicker scatterers (equations 2 and 3) has the difference that the scattering is represented as being generated by a combination of volume sources and perimeter sources. The shape of the scatterer here appears through defining a boundary to the region of sources. The use of equation (3) also has the advantage that appropriate phase relations are introduced between the “fictitious” sources in different parts of the obstacle through the terms \( P(k, z_2, \xi), \mathcal{B}'(\xi, \zeta) \).

**THE EFFECT OF SCATTERER THICKNESS**

As the thickness of a scatterer increases, there will be departures from the linear relation between scatterer thickness and scattered wave amplitude assumed in the simple thin scatterer theory. To examine the nature of these departures, we consider a slightly artificial case which has the advantage that various integrals can be evaluated analytically.

We consider a scatterer in the form of a lozenge (Figure 2) within which the elastic parameters are constant, and we adopt the thick scatterer approximation. We take a plane \( P \) wave incident at an angle \( \theta \) to the \( z \) axis and to make comparisons easier consider

![Fig. 2. Lozenge scatterer used in theoretical calculations.](image)

only the part of the scattered wave field with the same horizontal wave number as the incident wave \( k = \omega \sin \theta / \alpha \). Then for this scatterer, the thickness dependence has the form

\[
1/\omega^2 \cos^2 \theta/\alpha^2 \left\{ d_1 2\omega/\alpha \cos \theta \sin \left( \frac{\omega H}{\alpha} \cos \theta \right) - 2(d_2 - d_1)/H \cos \left( \frac{\omega H}{\alpha} \cos \theta \right) - 1 \right\}
\]

(6)
where $d_1$, $d_2$ are defined as in Figure 2. Expanding (6) in powers of $H$ up to third order we have

$$H \left\{ (d_1 + d_2) - \frac{1}{12} (3d_1 + d_2) \omega^2 H^2 \frac{\cos^2 \theta}{\lambda^2} \right\}. \quad (7)$$

Thus for small $H$ the scattered amplitude is linear in $H$ but departures occur as the thickness of the scatterer increases. In Figure 3 we have compared the results from (6) derived for thick scatterers with the linear relation derived from thin scatterer theory [this is the same as the first term in (7)]. We see that the agreement is best when $d_1/d_2$ is smallest and, thus, when the thin scatterer criterion of small-boundary slope is most closely satisfied.

![Figure 3](image-url)

**Fig. 3.** The dependence of the scattered amplitude (arbitrary units) on the thick scatterer approximation, for various values of $d_1/d_2$ as marked, compared with the linear prediction of thin scatterer theory.

As a rough rule the departures from linearity should not be more than 10 per cent provided

$$H < \frac{\Lambda \sec \theta}{10} \quad (8)$$
where $\Lambda_0$ is the wavelength of the $P$ wave. When this result is applied, the angle of incidence should not be too large ($\theta \leq 25^\circ$), since, for large $\theta$, the thickness dependence we have calculated will no longer be typical of the whole scattered wave field. Within the range specified by the relation (8), it is usually satisfactory to approximate the scattering effect of an obstacle by a single line of source elements rather than using a volume source distribution.

We may note that the thick scatterer formulation (6) predicts that the scattered amplitude should decrease with increasing scatterer thickness after a certain value of $H \cos \theta / \Lambda_0$ is reached; this effect is due to the interference of waves scattered by different parts of the scatterer. This prediction of first-order scattering theory is confirmed experimentally using the techniques of two-dimensional model seismology (O'Brien and Symes, 1971). For a lozenge scatterer ($\theta = 0^\circ$, $d_1/d_2 = 0.75$), the ratio of the scattered amplitudes, recorded at a position vertically above the center line of the scatterer, for lozenge thicknesses of 0.18 $\Lambda_0$, 0.36 $\Lambda_0$ is found to be 1.14; from the theoretical expression (6) we find the ratio to be 1.09 in good agreement with the measured value (on the linear thin scatterer theory this ratio would be 2.00).

We see, therefore, that in this example the departures from linearity in $H$ are well accounted for by the thick scatterer approximation. In many geophysical scattering problems the velocity contrasts between scatterer and matrix are not too severe ($\leq 15$ per cent), and, in these circumstances, it will often be quite a good approximation to use thick scatterer theory.

ACKNOWLEDGMENT

I would like to thank Dr. E. R. Lapwood for advice and encouragement, and the Chairman and Directors of the British Petroleum Company Limited for the award of a Research Studentship and permission to publish this paper.

REFERENCES

Kennett, B. L. N. (1972b). Seismic wave scattering by obstacles on interfaces, Geophys. J. 28, 249–266.