Rapid calculation of surface wave dispersion

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Summary. The secular equation for Love modes is that the phase of the SH reflection coefficient should be an integer multiple of $2\pi$. For a model consisting of uniform layers direct calculation of the reflection coefficient phase can be used as the basis of a fast, iterative calculation scheme for Love mode dispersion. At fixed slowness the mode branches are nearly regularly spaced in frequency and multimode calculations can be made very quickly.

For Rayleigh waves a comparable development in terms of the phase of the SV reflection coefficient can be made for moderately high frequencies, with an allowance for coupling to evanescent P-waves.

The iterative calculations for the dispersion are computationally stable and achieve very good accuracy at a fraction of the cost of a full treatment. For high-order Rayleigh modes the computational cost can be reduced by a factor of 20 or more. This should allow efficient modal synthesis of such phases as Lg, at high frequencies.

1 Introduction

Existing methods of calculating surface wave dispersion are very effective but involve substantial amounts of computation if many mode branches are to be delineated. At high frequencies special precautions have to be taken to avoid loss of precision problems associated with evanescent waves. These effects are most pronounced for Rayleigh waves but can be controlled by using a higher-order matrix system to calculate the secular function directly (Dunkin 1965; Harkrider 1964). Recently the construction of theoretical seismograms from a large number of modes has led to an improvement in the efficiency of dispersion calculations. Woodhouse (1981) describes a higher-order matrix system whilst Harvey (1981) uses a modification of a method due to Abo-Zena (1979) to find the secular function and then searches for roots in slowness at fixed frequency. This procedure is very efficient for the lowest-order modes but does not exploit the regularities in the spacing of higher modes in frequency. Kerry (1981) has used an approach based on the use of reflection matrices which circumvents the loss of precision problems. He reduces the computational cost of calculations in stacks of uniform layers by using stored interfacial reflection and transmission
coefficients at fixed slowness and thus isolates the frequency dependence in the phase terms. The root finding is also made easier by the nearly regular spacing in frequency.

In terms of reflection matrices the secular function takes the form (Kennett 1974)

$$\det [I - R^O_L(\omega, p) \tilde{R}(p)] = 0,$$

where $I$ is the unit matrix, $\tilde{R}$ the matrix of free surface reflection coefficients and $R^O_L$ the reflection matrix for the entire stratification in the half-space. For Rayleigh waves the matrices are $2 \times 2$ with the $PP$, $SP$, $PS$ and $SS$ reflection coefficients as their entries. For Love waves (1.1) reduces to a scalar equation. Kerry (1981) worked directly with (1.1) but the method developed in this paper works with the phase behaviour of the reflection coefficients.

For both Love and Rayleigh waves we work with the phase of the $S$-wave reflection coefficients. The secular function for Love waves has an exact representation in terms of the phase of the $SH$-wave reflection coefficient from the stratification. This can be used to set up an efficient iterative scheme for finding the roots of the secular equation in frequency, keeping the slowness fixed. For Rayleigh waves, in the slowness range for which $P$-waves are evanescent at the surface, we can produce an approximate secular relation in terms of the $SV$-wave reflection coefficient from the stratification (Kennett 1982; Kennett & Clarke 1983). With a correction to allow for coupling of $SV$- and evanescent $P$-waves we can achieve good results at moderate to high frequencies.

2 Love modes

For $SH$-waves we are dealing with a scalar wave system and the secular equation has a particularly simple form. Since the free surface reflection coefficient is unity (1.1) reduces to the requirement that the $SH$-wave reflection coefficient from the stratification satisfies

$$R^O_L(\omega, p) |_{HH} = 1.$$  

This Love wave secular equation is stratified for those frequency/slowness pairs for which a plane downgoing $SH$-wave can be reflected back by the structure in the half-space $z > 0$ without change of amplitude or phase.

For a given structure equation (2.1) may be used directly as the basis of a numerical root finding scheme. With a recursive construction for the reflection coefficient $R_D$ this leads to a procedure which parallels the development given by Kerry (1981).

An alternative viewpoint is, however, to view (2.1) as an equation for the phase $\chi_D$ of the reflection coefficient $R_D$:

$$\chi_D(\omega, p) = 2n\pi, \quad n = 0, 1, 2, \ldots$$

If we track the phase $\chi_D$ as a function of frequency and slowness, a Love mode occurs when $\chi_D$ is a multiple of $2\pi$. Each mode branch is tagged by the order number $n$ corresponding to the phase offset between an incident and reflected wave. Equation (2.2) may be thought of as a constructive interference condition for $SH$-waves multiply reflected, above at the free surface and below by the stratification.

This phase dispersion property is the basis of the differential equation for the phase used by Keilis-Borok, Neigauz & Shkadinskaya (1965). Tolstoy (1955) recognized that for scalar wave systems an efficient iterative scheme for calculating surface wave dispersion could be developed using phase calculations in multilayered media. The method we present here is an extension of Tolstoy’s work incorporating features of the asymptotic method used by Kennett & Nolet (1979) for calculations of high-frequency toroidal modes in piecewise continuous earth models.
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We will first illustrate the method with the familiar example of a layer over a half-space and then discuss the extension to a multilayered medium.

2.1 A LAYER OVER A HALF-SPACE

Consider a single layer with density $\rho_0$, shear wave speed $\beta_0$ and thickness $h_0$ overlying a uniform half-space with density $\rho_1$ and shear wave speed $\beta_1$. The interfacial reflection coefficient for slowness $p$ at $h_0$ is

$$r_B^1(p) = (\rho_0 \beta_0^2 q_{\beta_0} - \rho_1 \beta_1^2 q_{\beta_1})/(\rho_0 \beta_0^2 q_{\beta_0} + \rho_1 \beta_1^2 q_{\beta_1})$$

in terms of the vertical wave slownesses

$$q_{\beta_0} = (\beta_0^2 - p^2)^{1/2}, \quad q_{\beta_1} = (\beta_1^2 - p^2)^{1/2};$$

with a choice of branch cut for the radicals such that $\text{Im}(\omega q_{\beta_0}) > 0, \text{Im}(\omega q_{\beta_1}) > 0$.

In order to construct the reflection coefficient as seen at the surface $R^D_{DL}(\omega, p)$, we have to include the phase associated with propagation down to the level $h_0$ and back again. Thus equation (2.1) takes the form

$$\exp(2i\omega q_{\beta_0} h_0) r_B^1(p) = 1, \quad (2.4)$$

and roots of this secular equation, for real frequency, will only be possible when $r_B^1$ has unit modulus and $q_{\beta_0}$ is real. This means that we must have propagating waves in the upper layer and evanescent waves, with $q_{\beta_1}$ imaginary, in the half-space. The Love wave slowness is thus restricted to

$$\beta_1^{-1} < p < \beta_0^{-1}, \quad (2.5)$$

and the corresponding frequencies are determined by the phase condition (2.2)

$$2\omega q_{\beta_0} h_0 + \chi^1(p) = 2n\pi, \quad (2.6)$$

where $\chi^1$ is the phase of the interface coefficient $r_B^1$ (2.3). It is conventional to find the dispersion in this case from the relation

$$\tan \omega q_{\beta_0} h_0 = -i \rho_1 \beta_1^2 q_{\beta_1}/\rho_0 \beta_0^2 q_{\beta_0},$$

(see, e.g. Ewing, Jardetzky & Press 1957) which is obtainable by a rearrangement of (2.4). However, at fixed slowness, we may write (2.6) as

$$\omega_n(p) = [n\pi - \frac{1}{2} \chi^1(p)]/q_{\beta_0} h_0$$

so that multimode dispersion may be obtained directly. $\chi^1(p)$ decreases steadily with increasing $p$ approaching $-\pi$ when $p \to \beta_0^{-1}$ and is zero when $p = \beta_1^{-1}$. The entire dispersion curves are therefore traced out as the slowness varies across the range (2.5).

2.2 LOVE WAVES IN A MULTILAYERED MEDIUM

For a perfectly elastic half-space, we may extend the approach we have used for the single layer to deal with a multilayered stack, in which material with density $\rho_l$ and shear wave
speed $\beta_j$ lies between the interfaces $z_j$ and $z_{j+1}$. The reflection coefficient at $z = 0$ will have unit modulus when there are evanescent waves in the underlying half-space and travelling waves at the surface, that is for slownesses in the range

$$\beta_L^{-1} < p < \beta_0^{-1},$$

where $\beta_L$ is the shear wave speed in the half-space beneath the layering.

The reflection coefficient at the surface, $z = 0$, is related to that just above the first interface $z_1(= h_0)$ by the phase delay in the top layer as in (2.4), so that (2.1) becomes

$$\exp(2i\omega \beta_0 z_1) R_D(z_1-, \omega, p) = 1.$$  \hfill (2.8)

The reflection coefficient $R_D(z_1-)$ must still have modulus unity. Thus if we set $R_D(z_1-) = \exp(ix_1)$ we require

$$\omega_n \beta_\beta h_0 = n\pi - \frac{1}{2} x_1(\omega_n, p),$$  \hfill (2.9)

as in the treatment of the single layer case, but now the phase term depends on frequency.

For this SH-wave case, $R_D(z_1-)$ is related to the reflection coefficient just above the second interface $R_D(z_2-)$ and the interfacial coefficient $r_D^1$ at $z_1$ by

$$R_D(z_1-) = \exp(ix_1) = \frac{r_D^1 + R_D(z_2-) \exp(2i\omega \beta_\beta h_1)}{1 + r_D^1 R_D(z_2-) \exp(2i\omega \beta_\beta h_1)}.$$  \hfill (2.10)

with $h_1 = z_2 - z_1$. This form of the recursion relation for reflection (Kennett 1974) exploits the properties of the SH interface coefficients. We can write $r_D^1$ as

$$r_D^1 = (1 - Q^1)/(1 + Q^1),$$  \hfill (2.11)

where

$$Q^1 = \rho_1 \beta_\beta P_\beta /\rho_0 \beta_\beta q_\beta q_\beta;$$

and use the exponential representation of a tangent to find an expression for the phase $x_1$. Thus

$$\tan\left(\frac{1}{2} x_1\right) = -i Q^1 \frac{R_D(z_2-) \exp(2i\omega \beta_\beta h_1) - 1}{R_D(z_2-) \exp(2i\omega \beta_\beta h_1) + 1}.$$  \hfill (2.12)

When we have propagating SH-waves on both sides of the interface at $z_1$, $Q^1$ is real, and $R_D(z_2-)$ must also have modulus unity so that

$$\tan\left(\frac{1}{2} x_1\right) = |Q^1| \tan(\omega \beta_\beta h_1 + \frac{1}{2} x_2),$$  \hfill (2.13)

where $R_D(z_2-) = \exp(ix_2)$. With the aid of trigonometric identities (2.13) can be recast directly in terms of the phases

$$\frac{1}{2} x_1 = \omega \beta_\beta h_1 + \frac{1}{2} x_2 - \tan^{-1} \left( \frac{r_D^1 \sin(2\omega \beta_\beta h_1 + x_2)}{1 + r_D^1 \cos(2\omega \beta_\beta h_1 + x_2)} \right).$$  \hfill (2.14)

If there is an evanescent wave below $z_1$ then $R_D(z_2-)$ and $iQ^1$ are real and so

$$\tan\left(\frac{1}{2} x_1\right) = |Q^1| \frac{R_D(z_2-) \exp(-2\omega |q_\beta| h_1) - 1}{R_D(z_2-) \exp(-2\omega |q_\beta| h_1) + 1}.$$  \hfill (2.15)
At very high frequencies, the evanescent decay reduces the right-hand side of (2.15) to $-|Q'|$. Similar results hold for any deeper interfaces.

The relations (2.14) and (2.15) and their counterparts for other interfaces may be used to build an iterative method for calculating surface wave dispersion. For fixed slowness $p$ we set a trial frequency $\omega_n^t$ for the $n$th mode, and, starting at the base of the layering, use a recursion scheme as in (2.10) to carry the $SH$ reflection coefficient to successively higher interfaces in the region where we have evanescent waves. As soon as we reach an interface $z_p$ above which we have propagating waves we calculate the reflection coefficient phase $\chi_p$

$$\exp \left[ i \chi_p (\omega, p) \right] = R_D (z_p, -\omega, p)$$

via (2.15). For higher interfaces we increment the phase term via (2.14) as

$$\frac{1}{2} \chi_j = \omega q_j h_j + \frac{1}{2} \chi_{j+1} - \tan^{-1} \left( \frac{r_{j}^{D} \sin 2\nu_j}{1 + r_{j}^{D} \cos 2\nu_j} \right),$$

(2.16)

where

$$\nu_j = \omega q_j h_j + \frac{1}{2} \chi_{j+1},$$

for each interface up to $z_1$. The total phase at the interface $z_1$ is then

$$\frac{1}{2} \chi_1 (\omega, p) = \sum_{j=1}^{p-1} \omega q_j h_j + X_1 (\omega, p),$$

(2.17)

where

$$X_1 (\omega, p) = \sum_{j=1}^{p-1} \tan^{-1} \left( \frac{r_{j}^{D} \sin 2\nu_j}{1 + r_{j}^{D} \cos 2\nu_j} \right) + \chi_p (\omega, p).$$

(2.18)

When we are at the frequency of a mode the phase (2.17) will be such that

$$\frac{1}{2} \chi_1 (\omega_n, p) = n\pi - \omega_n q_{\beta_n} h_n.$$

Otherwise we may derive an improved estimate of the mode frequency $\omega_n' \left( 0 \right)$ from

$$\omega_n' \sum_{j=0}^{p-1} q_{\beta_j} h_j = n\pi - X_1 (\omega_n', p),$$

(2.19)

and then use this new estimate of the mode frequency to start the cycle of phase estimation once again. This iterative scheme works very effectively: a convenient starting frequency for the $n$th branch is

$$\omega_n^0 = n\pi \left( \sum_{j=0}^{p-1} q_{\beta_j} h_j \right),$$

and the process is terminated when two successive frequency estimates agree to within a stated accuracy. The actual number of iterations depends strongly on the structure and the accuracy which is demanded but, as an example, for an absolute accuracy of $10^{-6}$ in $\omega$ in the T7 model used in Section 4 between 3 and 9 iterations are normally required. Although it is clear from (2.19) that $|X_1| < \pi/2$, numerical difficulties can arise if the modulus of $X_1$ becomes very close to $\pi/2$. In the original method of Tolstoy (1955) the phase effects were calculated using (2.13), and evanescent contributions ignored. The present scheme gives a faster convergence for the modal frequency.
For the values of the slowness $p$ corresponding to a region of shear velocity inversion, the present scheme is suitable for estimating the dispersion of modes trapped in the crust, but not for those trapped in the low velocity channel. In this range it is most effective to use a full numerical calculation similar to that described by Kerry (1981) to get the full details of modal interference. Outside of this interval the iterative scheme may be used directly.

3 Rayleigh waves

The full secular function for Rayleigh waves (1,1) involves the determinant of a combination of the $2 \times 2$ reflection matrices, $\tilde{R}$ from the surface and $R_{D}^{OL}$ from the stratification. In terms of $P$- and $S$-wave reflection coefficients we have

$$1 - \tilde{R}^{SS}(R_{D}^{PP} + R_{D}^{SS}) - 2 \tilde{R}^{PS}R_{D}^{PS} + iR_{D}^{SS}R_{D}^{PP} - (R_{D}^{PS})^2 = 0,$$

(3.1)

where we have used the symmetry of the reflection matrices

$$\tilde{R}^{PS} = \tilde{R}^{SP}, \quad R_{D}^{SP} = R_{D}^{PS}$$

and the properties

$$\tilde{R}^{PP} = \tilde{R}^{SS}, \quad \det \tilde{R} = 1.$$

For slownesses such that $P$-waves are evanescent throughout the half-space but $S$-waves have travelling wave character at the surface, the reflection coefficients $R_{D}^{PP}$, $R_{D}^{PS}$ for high frequencies will be very small compared with $R_{D}^{SS}$. The dispersion relation (3.1) is therefore dominated by $S$-wave propagation and if all $P$ contributions could be neglected we obtain

$$R_{D}^{SS}(\omega, p) \tilde{R}^{SS}(p) = 1.$$

(3.2)

For a perfectly elastic medium, both $R_{D}^{SS}$ and $\tilde{R}^{SS}$ have modulus unity and so we have a comparable situation to the Love wave case.

When $P$-wave terms are included we can exploit the unitarity properties of the reflection coefficients to obtain a relatively simple expression for the secular equation (Kennett 1982). We set

$$R_{D}^{SS} = \exp(i\psi), \quad \tilde{R}^{SS} = \exp(i\psi_0),$$

(3.3a)

and extract the phase dependence of $R_{D}^{PP}$ as

$$R_{D}^{PP} = |R_{D}^{PP}| \exp(i\phi).$$

(3.3b)

The $PS$ reflection coefficient is also determined by these properties:

$$|R_{D}^{PS}|^2 = 2 \text{Im}(R_{D}^{PP}), \quad \arg R_{D}^{PS} = \pi/4 + \frac{1}{2}\psi.$$

(3.3c)

When these expressions for the reflection coefficients are substituted in (3.1), the secular equation may be expressed as

$$\sin \frac{1}{2}(\psi + \psi_0) + |R_{D}^{PP}| \sin \frac{1}{2}(\phi + \psi_0 - \psi) = |R_{D}^{PS}| |\tilde{R}^{PS}|.$$

(3.4)

For evanescent propagation, at even moderate frequency, $|R_{D}^{PP}|$ will be small, and so if we rewrite (3.4) as

$$\sin \frac{1}{2}(\psi + \psi_0) = |\tilde{R}^{PS}| [2 |R_{D}^{PP}| \sin \phi]^{1/2} - |R_{D}^{PP}| \sin \frac{1}{2}(\phi + \psi_0 - \psi),$$

(3.5)
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the first term on the right-hand side will dominate, but both will diminish with increasing frequency. A good approximation to the secular equation (3.5) is thus provided by

$$\psi(\omega, p) = 2n\pi - \psi_0(p) + 2 \sin^{-1} \{ |\tilde{R}_{PP}| [2 |R_{D}^{PP}| \sin \phi]^{1/2} \}.$$  (3.6)

Despite the correction for evanescent P and the slowness dependence of the phase of the \(SV\)-wave reflection coefficient at the free surface, (3.6) is of the same general form as (2.9). This suggests that we should once again be able to use an iterative technique to calculate the Rayleigh wave dispersion, by working with the phase of the \(SV\) reflection coefficient.

The basic calculation scheme is as described in Section 2 but some modifications need to be made. In the region where both \(P\) and \(S\)-waves are evanescent, it is normally sufficient to consider only the \(SV\) contribution to the reflection coefficient as

$$R_D(z_j -) = r_D^D + \frac{t_U^D t_D^U R_D(z_j+1 -) \exp \{-2\omega |q_{\beta_j}| h_j\}}{1 - r_U^D R_D(z_j +1 -) \exp \{-2\omega |q_{\beta_j}| h_j\}} \quad (3.7)$$

where \(r_D^D\), \(t_D^D\) are the SS reflection and transmission coefficients at the \(j\)th interface (the indices D and U refer to the direction of propagation of the corresponding incident waves). The relations between reflection and transmission coefficients are not quite as in the scalar case, and so all the coefficients appear explicitly. When we reach the interface \(z_p\) we evaluate the starting phase \(\psi_p\) from

$$\exp \{i\psi_p(\omega, p)\} = R_D(z_p-, \omega, p).$$

The difference between the \(SH\) and \(SV\)-wave cases in the propagating regime is that the interfacial reflection coefficients \(r_D^U\), \(r_U^D\) are no longer real, and do not satisfy the relation \(r_D^U = -r_U^D\). In addition, it is necessary to introduce the quantity

$$I_{\alpha}^D = t_U^D t_D^D - r_U^D r_D^D,$$

which in the \(SH\) case is identically equal to one. For the \(SV\)-wave case we may use an incremental phase development analogous to (2.16), which for \(z_j < z_p\) takes the form

$$\frac{1}{2} \psi_j = \omega q_{\beta_j} h_j + \frac{1}{2} \psi_{j+1} + \frac{1}{2} \arg \left\{ \frac{\exp(-2iu_j) r_D^j + I_{\alpha}^j}{1 - \exp(2iu_j) r_U^j} \right\} \quad (3.8)$$

with

$$u_j = \omega q_{\beta_j} h_j + \frac{1}{2} \psi_{j+1}.$$  

We carry this calculation up to the level \(z_1\), and the resulting surface phase is thus approximated by

$$\frac{1}{2} \psi(\omega, p) = \sum_{j=0}^{p-1} \omega q_{\beta_j} h_j + \Psi(\omega, p), \quad (3.9)$$

where now

$$\Psi(\omega, p) = -\frac{1}{2} \sum_{j=1}^{p-1} \arg \left\{ \frac{\exp(-2iu_j) r_D^j + I_{\alpha}^j}{1 - \exp(2iu_j) r_U^j} \right\} + \psi_p(\omega, p). \quad (3.10)$$

Corrections to \(\Psi(\omega, p)\) to allow for \(PS\) conversion are necessary if there are major interfaces close to the surface. The evanescent \(P\)-wave contribution arises from the near surface region.
and can be evaluated by using (3.7) but now with PP reflection and transmission coefficients.

With a trial frequency $\omega_n'$ for the $n$th higher Rayleigh mode we may derive an improved estimate of the mode frequency by combining the results of (3.6) and (3.9) to give

$$\omega_n^{-1} \sum_{j=0}^{p-1} q_{ji} h_j = n\pi - \frac{1}{2} \psi_0(p) + 2 \sin^{-1} \left\{ \frac{R_{PS}^p}{2 |R_{PS}^p| \sin \phi} \right\} . \quad (3.11)$$

The iteration over frequency is then repeated until successive frequency estimates agree to within some prescribed tolerance, the convergence being usually a little slower than for the Love wave.

The scheme we have just described works very well once the P-waves are strongly evanescent, but is inadequate at low frequencies. Thus if (3.11) is used as it stands for the fundamental and first higher Rayleigh modes the qualitative behaviour will be correct but the further correction for coupling between $P$ and $S$ which appears in (3.5) must be added to the right-hand side of (3.11).

4 Computational results

In this section we illustrate the computational characteristics of the method we have just described, by comparison with high precision numerical calculations using the algorithms of Kerry (1981).

The calculations have been performed for the uniform layer representation of the model T7 (Burdick & Helmberger 1978) illustrated in Fig. 1. In order to keep the iterative development as simple as possible we have included an evanescent contribution to the starting phase

![Figure 1. The model T7.](http://gji.oxfordjournals.org/ at Australian National University on February 19, 2013)
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Figure 2. Rayleigh wave dispersion for model T7.

\( \chi_p \) or \( \psi_p \) from just the layer beneath \( z_p \). With the layer thickness in the model of around 10 km this will normally be sufficient for frequencies above about 0.3 Hz. For Rayleigh waves we included an evanescent \( P \) correction in the first layer of the crust.

In Fig. 2 we display the dispersion curves for all mode branches for Rayleigh waves in the slowness range 0.15 - 0.32 s km\(^{-1}\) for frequencies up to 0.95 Hz, calculated using the technique of Kerry (1981). At fixed slowness we can see the nearly regular spacing of mode branches discussed in Section 3. Within the slowness range \( \alpha_0^{-1} < \rho < \beta_0^{-1} \) indicated in Fig. 2 we can use the iterative technique to calculate the dispersion of the modes. Except in the region corresponding to the velocity inversion marked in Fig. 2 as 'ivz', the iterative treatment generates dispersion curves which closely reproduce those of Fig. 2 over the entire frequency range. In the zone 'ivz' we get interference between waves trapped in the crust and those trapped in the velocity inversion as discussed in detail by Kerry (1981); this behaviour cannot be reproduced unless allowance is made throughout the layering. For frequencies greater than 0.5 Hz the iterative calculations visually overlay those shown in Fig. 2 outside the region 'ivz'. The Love wave dispersion for T7 is very similar to Fig. 2 for slownesses greater than \( \alpha_0^{-1} \).

In Fig. 3 we present a numerical comparison of iterative and complete dispersion calculations interpolated to a frequency of 0.95 Hz, that is in a slice taken across the top of Fig. 2. The iterative calculations were carried out in single precision on an IBM 370/165 with a convergence criterion of \( 10^{-6} \) difference between successive estimates of \( \omega \). About 7 - 10 iterations were needed for the rapid method in most cases. The complete calculations in double precision on the same machine should be accurate to at least nine decimal places. Outside the region 'ivz' the percentage error in slowness \( (\Delta \rho/\rho) \) is very small and the iterative techniques are seen to work very well. Both for Love modes, indicated by the open symbols and for Rayleigh modes, indicated by stars, the error fluctuates about zero with values typically less than 0.025 per cent, which at a phase velocity of 4 km s\(^{-1}\) would be an error of 0.001 km s\(^{-1}\). Considered as a percentage of the mean model spacing the average
Figure 3. Comparison of iterative results for Love and Rayleigh wave dispersion and full numerical calculations at 0.95 Hz. Love modes are indicated by open symbols, Rayleigh modes by stars.

error over all the modes is about 0.5 per cent for both Love and Rayleigh modes. We would expect there to be greater error in the Rayleigh wave case because of the number of approximations made in constructing the iterative scheme.

With a broad range of slownesses the sheer numbers of modes needed at high frequency normally places a limit on the highest frequency actually used in the computation. To generate all the Rayleigh mode curves shown in Fig. 2 required about 3 hr of computation time.

Figure 4. Surface wave dispersion in model T7 at higher frequencies. Rayleigh modes are indicated by heavier lines.
on an IBM 370/165. The corresponding iterative calculations, whose accuracy in fact increases with increasing frequency, took less than 2 min computation time. Slightly less time than this is required for the Love mode calculations, but there the complete calculations are also cheaper.

To illustrate the power of the iterative technique at high frequencies we take the same model, T7, but consider crustal propagation up to 6 Hz. Both Love and Rayleigh mode dispersion curves are plotted in Fig. 4 and the similarities are very striking; it is this feature which is exploited in the iterative calculation schemes. The offsets between the Love and Rayleigh wave mode branches are largely due to the phase effect of the free surface reflection for SV-waves. We once again compare iterative and complete calculations at fixed frequency, now 4 Hz, in Fig. 5. The errors in slowness are now, in general, much smaller than in Fig. 3, as we would expect.

![Percentage Error in Phase Slowness](image)

Figure 5. Comparison of iterative results for Love and Rayleigh wave dispersion and full numerical calculations at 4 Hz. Love modes are indicated by open symbols, Rayleigh modes by stars.

For both Love and Rayleigh waves the error is negligible over most of the slowness range, with a few errors of up to 0.05 per cent, and once again the average, over all modes, of the error as a percentage of modal spacing is less than 0.5 per cent.

The largest errors for both Love and Rayleigh waves occur at slownesses such that $S$-waves are just evanescent in the second layer. In this case the decay across the second layer is so weak that there is significant contribution from deeper structure (even at 4 Hz) which has been ignored in our implementation of the iterative scheme. For these higher frequencies the relative computational speeds of the iterative and complete methods were around 100 to 1 for Rayleigh waves and 30 to 1 for Love waves.

Much of the saving results from the fact that the phase behaviour is explicitly represented and so the central core of the algorithm is extremely simple. This simplicity allows very efficient computer coding of the central loops and so enhances the speed relative to a general purpose program.
6 Discussion

We have shown that an iterative calculation scheme based on the phase of S-wave reflection coefficients provides an efficient and accurate method of calculating dispersion curves at high frequencies and large slownesses where propagation is dominated by S propagation. This will also allow further investigation of some seismic phases for which theoretical seismogram calculation by model residual sums has hitherto been prohibitively expensive. In particular it should be possible to examine the dependence of Lg and Sn on structure in the crust and the uppermost part of the mantle.

Even for the low-frequency regime where the dispersion calculations using the recursive method are of only moderate accuracy, these results can be of considerable utility. The rate of convergence of numerical root finding schemes depends on the accuracy of initial estimates of the roots. The iterative results provide a convenient starting point since they include the main features of the dispersion.

The method we have just described is restricted to the range of slownesses where P-waves are evanescent at the surface and throughout the half-space. When the slowness is only slightly less than the P-wave slowness at the surface, an additional approximate term can be introduced in (3.11) to allow for propagating P-waves but this is inadequate if significant P to S conversion occurs near the surface. In general once P-waves are propagating, full calculation methods such as that of Kerry (1981) will need to be used. This will be the case for geophysical exploration problems where very low wavespeed weathered layers occur at the surface. The 'locked-mode' method introduced by Harvey (1981) synthesizes P-waves via modal residue summation, and again the rapid calculation scheme will be useful only for slownesses larger than that for P-waves at the surface.

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References


