Seismic waves in a stratified half-space — IV: 
P–SV wave decoupling and surface wave dispersion

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Summary. In a region of smoothly varying seismic parameters, the seismic displacements can be transformed into portions depending principally on the P and SV parts of the wavefield. Although this representation has previously been used for body wave studies, it is also very suitable for understanding surface wave problems. With the aid of this transformation and the representation of Rayleigh wave dispersion in terms of reflection coefficients at the free-surface and from the stratification the behaviour of the Rayleigh wavefield with slowness can be related to the velocity structure. This allows the character of Rayleigh waves to be related to that of Love waves. There is very close correspondence when P-waves are evanescent throughout the stratification. In this case we can represent the Rayleigh wave behaviour in terms of a scalar eigenfunction which has simple, though approximate, orthogonality properties.

1 Introduction

In the earlier papers in this series (Paper I — Kennett & Kerry 1979; Paper II — Kennett 1980; Paper III — Kennett & Illingworth 1981) we have shown how the response of a stratified medium can be constructed in the frequency/slowness domain. Theoretical seismograms can then be formed by inversion of the integral transforms over frequency and slowness.

We have hitherto concentrated on the representation of the seismic response in terms of the reflection and transmission properties of the stratification since this provides a physically appealing and computationally effective description of the behaviour. Here we will be concerned with the circumstances in which it is possible to separate P- and SV-wave propagation in realistic earth models and in particular with the nature of surface wavetrains.

Although many methods have been devised to handle coupled P–SV-wave propagation in stratified media, based mostly on matrix techniques (see, e.g. Haskell 1953; Gilbert & Backus 1966; Takeuchi & Saito 1972; Kennett 1974, 1981); the majority of seismic interpretation is based on an assumption of weak coupling between P and SV except at the Earth’s surface. Richards (1974) attempted to quantify the level of coupling in stratified earth models, using an approximate development in terms of potentials, and showed that the
coupling terms diminish rapidly with increasing frequency. This result was used by Chapman (1974) and Woodhouse (1978) to produce an asymptotic representation of wave propagation in smoothly varying media. They used a sequence of variable changes to reduce the elastodynamic equations to a form where known asymptotic properties could be used. For piecewise smooth media Kennett & Illingworth (1981) have extended this work with a reflection matrix treatment. Starting with an approximate solution which does not allow for coupling between $P$- and $SV$-waves they show how coupling can be included by the iterative construction of correction terms, or the solution of a set of matrix Ricatti equations.

In this paper we propose an alternative viewpoint on seismic wave propagation in piecewise smooth media. We make a linear transformation of the stress and displacement variables and generate a new set of coupled differential equations which emphasize the $P$- and $S$-wavefields. This decomposition of the seismic wavefield is then used to examine Rayleigh wave dispersion, using comparisons with results derived from reflection matrix methods introduced in the earlier papers. We are able to make a number of connections between results derived previously by very different methods and illustrate the physical character of Rayleigh wave propagation. For large slowness Rayleigh waves we are able to find an approximate scalar eigenfunction with $SV$-wave character with simple orthogonality results.

2 Linear transformation of stress-displacement variables

For a horizontally stratified medium it is very convenient to make an expansion of the seismic wavefield in terms of vector surface harmonics in the horizontal variables (see, e.g. Paper I, section 2.1) which depend on frequency $\omega$ and slowness $p$. The coefficients of the harmonic expansions for displacement $(U, V, W)$ and traction $(P, S, T)$ then satisfy coupled ordinary differential equations in the depth variable $z$. For isotropic media these equations separate into two sets. The first set which links $U, V, P, S$ describes $P$–$SV$-wave propagation and in a source-free region takes the form

$$\frac{\partial}{\partial z} \begin{pmatrix} U \\ V \\ P \\ S \end{pmatrix} = \omega \begin{pmatrix} 0 & p(1-2\beta^2/\alpha^2) & \rho \alpha^2 & 0 \\ -p & 0 & 0 & \rho \beta^2 \\ -\rho & 0 & 0 & p \\ 0 & \rho(\nu p^2-1) & -p(1-2\beta^2/\alpha^2) & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ P \\ S \end{pmatrix}$$

where

$$\nu = 4\beta^2(1-\beta^2/\alpha^2).$$

The second set is somewhat simpler and is confined to $SH$-wave propagation

$$\frac{\partial}{\partial z} \begin{pmatrix} W \\ T \end{pmatrix} = \omega \begin{pmatrix} 0 & \rho \beta^2 \\ p(\beta^2 p^2-1) & 0 \end{pmatrix} \begin{pmatrix} W \\ T \end{pmatrix}.$$

A comparable structure exists for transversely isotropic medium with the symmetry axis vertical although the coefficients in the equations differ.

The behaviour of the equations (2.1), (2.2) can be summarized by writing them in the form

$$\partial_z b = \omega A(p, z) b$$

in terms of a stress-displacement vector $b$ which has the property that it is continuous across all planes $z = \text{constant}$. Seismic sources may be introduced into this scheme by including a
forcing term in (2.3). For a point source the effect is to introduce a jump in the vector \( \mathbf{b} \) across the source level \( z_s \).

To try to separate the coupling of \( P \)- and \( SV \)-waves we seek some transformation of variables such that the dominant dependence for each of \( P \) and \( S \) contributions resembles (2.3). As pointed out by Chapman (1974) this can be achieved by a variety of transformations. A convenient form is obtained by working with the new variables

\[
\begin{align*}
    u &= \frac{P}{\rho \alpha^2} - 2p \beta^2 \frac{\mathbf{a}^2}{\alpha^2} V, \\
    s &= \rho (2\beta^2 p^2 - 1) U - pS, \\
    w &= \frac{S}{\rho \beta^2} - 2pU, \\
    t &= \rho (2\beta^2 p^2 - 1) V - pP,
\end{align*}
\]

(2.4)

for which the inverse relations are also simple

\[
\begin{align*}
    U &= \rho (\beta^2 w) - s/\rho, \\
    V &= \rho (\alpha^2 u) - t/\rho, \\
    P &= -2p (\alpha^2 t) + \rho (1 - 2\beta^2 p^2) \alpha^2 u, \\
    S &= -2p (\beta^2 s) + \rho (1 - 2\beta^2 p^2) \beta^2 w.
\end{align*}
\]

(2.5)

The differential equations satisfied by these new variables are still coupled but can be cast into a form where the frequency terms appear in a block diagonal form partitioned by wave type. In terms of the vector

\[
c = [u, s, w, t]^T,
\]

(2.6)

where \( ^T \) denotes a transpose, the new differential equations take the form

\[
\partial_z c = \{ \omega \mathbf{F}_0 + \mathbf{\Gamma} \} c,
\]

(2.7)

where \( \mathbf{F}_0 \) is the block diagonal matrix

\[
\mathbf{F}_0 = \begin{pmatrix}
    0 & (\rho \alpha^2)^{-1} & 0 \\
    \rho (\alpha^2 p^2 - 1) & 0 & (\rho \beta^2)^{-1} \\
    0 & 0 & \rho (\beta^2 p^2 - 1)
\end{pmatrix}
\]

(2.8)

so that this portion of the coefficient matrix mirrors the behaviour in (2.2). The remaining portion of the coefficient matrix \( \mathbf{\Gamma} \) depends on the gradients in seismic parameters

\[
\mathbf{\Gamma} = \begin{pmatrix}
    (2\beta^2 p^2 - 1) \gamma_\alpha + 2p^2 \alpha^2 \partial_z (\beta^2/\alpha^2) & 0 & 2p^{-1} \left[ \gamma_\alpha + \partial_z (\beta^2/\alpha^2) \right] \\
    0 & 2\beta^2 p^2 \gamma_\beta & -2p^3 \beta^4 \gamma_\beta & 0 \\
    0 & 2p^{-1} \gamma_\beta & (2\beta^2 p^2 - 1) \gamma_\beta & 0 \\
    -2p^3 \beta^2 \alpha^2 \gamma_\beta & 0 & 0 & 2\beta^2 p^2 \gamma_\beta
\end{pmatrix}
\]

(2.9)

where

\[
\gamma_\alpha = \frac{\partial_z \rho}{\rho} + 2 \frac{\partial_z \alpha}{\alpha},
\]

\[
\gamma_\beta = \frac{\partial_z \rho}{\rho} + 2 \frac{\partial_z \beta}{\beta}.
\]
Within a region of smoothly varying seismic properties the quantities \( u, w, s, t \) are continuous. At a change in parameter gradients \( c \) is continuous although the coefficients \( \Gamma \) in the differential equation will be different on the two sides of this level.

In uniform media \( \Gamma \equiv 0 \) and so \( P \)- and \( SV \)-wave contributions propagate independently and this property is shared by the leading order term of an asymptotic solution of (2.7) (Woodhouse 1978), within a gradient zone. At a change in parameter gradient the change in the character of \( F_0 \) gives the main contribution to wave reflection but the change in \( \Gamma \) gives additional effects with weak coupling of \( P \)- and \( S \)-waves which become more significant at low frequencies.

At a discontinuity in seismic properties \( U, V, P, S \) are continuous but the new wave-related variables \( u, s, w, t \) are not continuous since their definition involves the elastic parameters. To cross a discontinuity we have therefore to construct \( U, V, P, S \) from the wave-related variables on one side of the interface via (2.5) and then construct the new forms of \( u, s, w, t \) on the other side of the interface using the appropriate elastic parameters (2.4). Since each of \( U, V, P, S \) involves a variable from both the \( P \)- and \( SV \)-wave groups we see that there must be \( P-SV \) coupling at an interface. At small slownesses \( V, S \) largely depend on \( (w, t) \) and \( U, P \) on \( (u, s) \), so that we get the well-known weak coupling between \( P \)- and \( SV \)-waves at near vertical incidence.

Most previous work using the high-frequency decoupling of \( P \) and \( SV \) has concentrated on body wave problems, although Cormier (1980) has used some of the results in constructing complete theoretical seismograms in radially inhomogeneous media. However, the decoupling properties can become particularly important in surface wave dispersion when one wave type becomes evanescent.

3 Rayleigh wave dispersion

In a Rayleigh mode the seismic wavefield has to satisfy the dual conditions of vanishing traction at the surface of the stratification and decay of both horizontal and vertical displacements at \( z \rightarrow \infty \). As a result the bulk of the energy is confined in the upper part of the stratification. The \( P \) and \( SV \)-wave components in the wavefield are coupled through both the upper and lower boundary conditions. This is well illustrated by the reflection matrix representation of the secular equation for Rayleigh waves (Paper I - 5.25).

For frequency \( \omega \) and slowness \( p \) we introduce the \( 2 \times 2 \) reflection matrix \( \tilde{R}(p) \) whose entries are the \( PP, PS, SP, SS \) reflection coefficients at the free surface. With a normalization of \( P \)- and \( S \)-wave components with respect to energy flux (Paper I - 2.16) the matrix \( \tilde{R} \) is symmetric and

\[
\tilde{R}^{PP} = \tilde{R}^{SS}, \quad \tilde{R}^{PS} = \tilde{R}^{SP}, \quad \text{det} \, \tilde{R} = 1. \tag{3.1}
\]

The reflection of \( P \)- and \( SV \)-waves incident from above on the stratification is represented by the \( 2 \times 2 \) matrix \( R^{OL}_D(\omega, p) \). The frequency-dependent entries in \( R^{OL}_D \) are the \( PP, PS, SP, SS \) plane wave reflection coefficients from the entire half-space at slowness \( p \). \( R^{OL}_D(\omega, p) \) will also be symmetric. These two reflection matrices incorporate the appropriate boundary conditions, the vanishing traction condition determines the coefficients \( \tilde{R} \) and \( R^{OL}_D \) is constructed to give outgoing or decaying wavefields in \( z > z_L \) (where \( z_L \) is the level at which the stratification is underlain by a uniform half-space). In terms of these reflection matrices the secular function for Rayleigh waves takes the form

\[
\text{det} \left[ I - \tilde{R}(p) R^{OL}_D(\omega, p) \right] = 0, \tag{3.2}
\]

which may be viewed as a constructive interference condition for waves successively
reflected from the free surface and the stratification. The explicit form of the determinant can be reduced to

$$\tilde{R}^{SS}(R_D^{PP} + R_D^{SS}) + 2\tilde{R}^{PS} R_D^{PS} - [R_D^{PP} R_D^{SS} - (R_D^{PS})^2] = 1,$$

(3.3)
on using the properties of $\tilde{R}$ (3.1) and the symmetry of $R_D^{OL}$.

We can get an alternative expression for the secular function by working in terms of the vector $c$ introduced in Section 2. The surface traction condition requires $P(0)$ and $S(0)$ to vanish and so from (2.5)

$$\rho_0 (1 - 2\beta_0^2 p^2) u(0) - 2p t(0) = 0,$$

$$\rho_0 (1 - 2\beta_0^2 p^2) w(0) - 2p s(0) = 0.$$  
(3.4)

These two equations may be combined into the single condition

$$\xi^2 \frac{s(0)}{\rho_0 \beta_0^2 u(0)} \cdot \frac{t(0)}{\rho_0 \beta_0^2 w(0)} = 1,$$

(3.5)

where

$$\xi = 2p \beta_0^2/(1 - 2\beta_0^2 p^2).$$

We recall that the variables $(u, s)$ are associated with $P$-waves, and $(w, t)$ with $SV$-waves, and so we set

$$y_\alpha = \frac{s(0)}{\rho_0 \beta_0^2 u(0)},$$

$$y_\beta = \frac{t(0)}{\rho_0 \beta_0^2 w(0)}.$$  
(3.6)

We can write the condition (3.5) on the $P$ and $S$ fields at the surface as

$$\xi^2 y_\alpha y_\beta = 1,$$

(3.7)

with the additional requirement that the fields $u, s, w, t$ appearing in (3.7) decay as $z \to \infty$.

### 3.1 Smooth Stratification

For the simple theoretical model of a smoothly stratified half-space, $y_\alpha$ and $y_\beta$ can be interpreted directly as the $P$ and $S$ contributions to the dispersion linked through the surface boundary condition. We see from (2.7) that the major contribution at high frequency $w F_0$ separates into $P$ and $S$ parts and does not depend on the parameter gradients which appear only in $\Gamma$. For each of the $P$- and $S$-wave parts we can match the dominant behaviour with a uniform Langer approximation based on Airy functions. Corrections to allow for parameter gradients may be made via an asymptotic expansion (Woodhouse 1978) or by an 'interaction series' of reflection terms (Kennett & Illingworth 1981).

For this simple model, in the leading order approximation we can take over the results of Kennett & Woodhouse (1978) for spheroidal modes on radial stratification. The condition of decaying displacement at depth requires a solution in terms of $Ai(x)$, which has cosinusoidal behaviour for negative arguments and exponential decay for positive arguments. This behaviour is imposed separately for the $P$- and $S$-wave terms, with conversion between wave types allowed only at the surface. For the $P$-wave term

$$y_\alpha \sim \omega^{-1/3} |\partial_z \Phi_\alpha|_0 Ai'(-\omega^{2/3} \Phi_\alpha)/Ai(-\omega^{2/3} \Phi_\alpha),$$

(3.8)
where

\[ \Phi_\alpha = \text{sgn}(q_\alpha^2) \left\{ \frac{3}{2} \int_0^{Z_\alpha} |q_\alpha| \, dz \right\}^{2/3}, \quad q_\alpha^2 = \alpha^{-2} - p^2, \]

and the level \( Z_\alpha \) corresponds to a turning point \((q_\alpha = 0)\), if one is present. We have a comparable representation for \( y_\beta \). The behaviour of the terms \( y_\alpha, y_\beta \) depends on the nature of the wavefield. This is most easily seen by using the asymptotic representations for the Airy functions. When \( P \)-waves are propagating at the surface

\[ y_\alpha \sim q_{\alpha 0} \tan \left( \omega \int_0^{Z_\alpha} |q_\alpha| \, dz - \pi/4 \right), \quad p < \alpha_0^{-1}; \]  \tag{3.9}

but if \( P \)-waves are evanescent

\[ y_\alpha \sim |q_{\alpha 0}|, \quad p > \alpha_0^{-1} \]  \tag{3.10}

since the exponential decay factors cancel.

Provided that \( R_{D}^{PP} \) is negligible we can find an alternative representation of \( y_\alpha, y_\beta \) in terms of the \( PP \) and \( SS \) reflection coefficients from the stratification

\[ y_\alpha \sim i q_{\alpha 0} (1 - R_D^{PP})/(1 + R_D^{PP}), \]

\[ y_\beta \sim i q_{\beta 0} (1 - R_D^{SS})/(1 + R_D^{SS}). \]  \tag{3.11}

If we can use the asymptotic representations of the reflection coefficients from a gradient zone \( (\text{Paper III} - 3.30) \) we recover \( (3.9), (3.10) \).

With our assumption of \( P \) to \( S \) conversion only at the surface the secular equation \( (3.3) \) reduces to

\[ \tilde{R}_{SS}(R_D^{PP} + R_D^{SS}) - R_D^{PP} R_D^{SS} = 1, \]  \tag{3.12}

and the interlacing of the previous results comes from the representation of the \( SS \) free-surface reflection coefficient

\[ \tilde{R}_{SS} = \frac{1 - \xi^2 q_{\alpha 0} q_{\beta 0}}{1 + \xi^2 q_{\alpha 0} q_{\beta 0}}. \]  \tag{3.13}

3.1.1 \( P \)-waves propagating at the surface

The asymptotic form of the Rayleigh wave secular function from \( (3.7), (3.9) \) \( (\text{cf. Kennett} \& \text{Woodhouse 1978}; \text{Brokski}i \& \text{Levshin 1979}) \)

\[ \tan \left( \omega \int_0^{Z_\alpha} |q_\alpha| \, dz - \pi/4 \right) \tan \left( \omega \int_0^{Z_\beta} |q_\beta| \, dz - \pi/4 \right) = (q_{\alpha 0} q_{\beta 0} \xi^2)^{-1}, \]  \tag{3.14}

which may alternatively be cast into a form in which \( \tilde{R}_{SS} \) appears explicitly

\[ \sin \left\{ \frac{\omega}{2} \left( \int_0^{Z_\alpha} |q_\alpha| \, dz + \int_0^{Z_\beta} |q_\beta| |dz| \right) \right\} + \tilde{R}_{SS} \cos \left\{ \frac{\omega}{2} \left( \int_0^{Z_\alpha} |q_\alpha| \, dz - \int_0^{Z_\beta} |q_\beta| \, dz \right) \right\} = 0. \]

For slowness \( p \), the turning point for \( S \)-waves will occur at much greater depth than that for
and so the dominant phase terms in (3.14) will be those associated with S-wave propagation. This means that the solution of (3.14) for the frequency of the \( n \)th mode can be written as (Levshin 1981)

\[
2\omega_n \int_0^{Z_\beta} |q_\beta| \, dz = (2n + \frac{1}{2})\pi - \arg A_S(\omega_n, p) \tag{3.15}
\]

where \( A_S \) represents those free surface reflection effects connecting incident and reflected \( SV \)-waves. This includes simple reflection \((\tilde{R}^{SS})\) and also conversion at the surface to \( P \), followed by reflection at the surface and from the near surface portion of the stratification, and then eventual reconversion at the surface to \( SV \). This term \( A_S \) may be represented as a reflection series, allowing for higher order \( P \) multiples. Thus

\[
A_S = \tilde{R}^{SS} + \tilde{R}^{SP} R_D^{PP} \tilde{R}^{PS} + \tilde{R}^{SP} R_D^{PP} \tilde{R}^{PP} R_D^{PP} \tilde{R}^{PS} + \ldots
\]

\[
= \tilde{R}^{SS} + \frac{\tilde{R}^{SP} R_D^{PP} \tilde{R}^{PS}}{1 - \tilde{R}^{PP} R_D^{PP}}, \tag{3.16}
\]

and with the use of (3.1)

\[
A_S = (\tilde{R}^{SS} - R_D^{PP})/(1 - \tilde{R}^{SS} R_D^{PP}).
\]

If we use the asymptotic representation of \( R_D^{PP} \),

\[
R_D^{PP} \sim \exp \left( 2i\omega \int_0^{Z_\alpha} |q_\alpha| \, dz - i\pi/2 \right),
\]

we can recover (3.14). We can regard (3.15) as a constructive interference condition for \( SV \)-waves. The phase of the \( SV \)-wave reflected back from the smoothly varying medium must match with the cumulative phase of all the \( SV \) interactions with the surface, including conversion, which is represented by the term \( A_S \).

For Love waves the secular relation is just \( R_D^{HH} = 1 \) and for this smoothly varying model we have the asymptotic representation

\[
2\omega_n \int_0^{Z_\beta} |q_\beta| \, dz = (2n + \frac{1}{2})\pi. \tag{3.17}
\]

Higher mode Rayleigh wave dispersion is therefore similar to that for Love waves but the frequencies are shifted by the amount

\[-\frac{1}{2} \arg A_S(\omega_n, p) \int_0^{Z_\beta} |q_\beta| \, dz.\]

The effect of the \( P \) contribution is to superimpose a periodic disturbance on the frequencies of the modal branches which varies with slowness. This 'solotone' effect introduces modal pinch effects in which the spacing of the mode branches varies significantly, for which there is no correspondence in the Love wave dispersion.

### 3.1.2 P-waves evanescent at the surface \((\alpha_0^{-1} < \beta < \beta_0^{-1})\)

In this slowness regime we may still use the representation (3.9) for \( y_\beta \), but for the evanescent \( P \)-waves we determine \( y_\alpha \) from (3.10). The secular equation (3.7) is therefore
asymptotically of the form (Brodskii & Levshin 1979)

$$\tan \left( \omega \int_0^z |q_\beta| \, dz - \pi/4 \right) = (|q_\alpha 0| q_\beta 0 z^2)^{-1}. \quad (3.18)$$

The $n$th Rayleigh mode frequency is thus given by

$$2 \omega_n \int_0^z |q_\beta| \, dz = (2n + \frac{1}{2}) \pi - \arg \tilde{R}^{SS}, \quad (3.19)$$

and the only difference from the Love wave case is the phase of the $SS$ free-surface reflection coefficient.

The behaviour of $SV$-wave elements $(w, t)$ in the vector $c$ is also quite close to that for the displacement and stress quantities $(W, T)$ for $SH$-waves. The eigenfunction for the $n$th Love mode is just a multiple of the $Ai$ function and so

$$W \sim \rho^{-1/2} \beta_0^{-1} q_\beta^{-1/2} \cos \left\{ \omega_n \int_0^z |q_\beta| \, dz - \pi/4 \right\}. \quad (3.20)$$

The solution for the variable $w$ corresponding to the $n$th higher Rayleigh mode is

$$w \sim \rho^{-1/2} \beta_0^{-2} q_\beta^{-1/2} \cos \left\{ \omega_n \int_0^z |q_\beta| \, dz - \pi/4 \right\}. \quad (3.21)$$

To the leading order approximation, we can achieve exact correspondence between the $SH$ and $SV$ results by working with the new variables $\beta w, \beta t$.

3.2 General Stratification

The simplicity of the preceding results is attractive and they may be used, to fair accuracy, at moderate frequencies when the shear wavelength is long enough to blur out structural details in the model (Kennett & Woodhouse 1978). Once we consider realistic earth models we have to take account of the effect of interfaces and to allow for possible $S$ to $P$ conversion within the stratification.

The surface condition (3.7) and the requirement that $u, w, s, t$ should decay as $z \to \infty$ are sufficient to determine Rayleigh wave dispersion for a general model. But the simple interpretation of $y_\alpha$ and $y_\beta$ in terms of separate $P$- and $S$-wave contributions we have made in Section 3.1 is no longer tenable. Although at high frequency we have nearly independent propagation of the $(u, s)$ and $(w, t)$ contributions to the seismic field within a gradient zone, once we have a discontinuity in elastic parameters in the model the elements of the vector $c$ are coupled. We have to work in terms of the original stress-displacement vector $b$ to handle the interface conditions, thereby mixing the $P$ and $S$ wavefields in $(u, s)$ and $(w, t)$ and giving non-zero $PS$ reflection coefficients.

The significant of the $P$-$S$ coupling depends on the nature of the wavefields on the sides of the interface. Even a major discontinuity deep in a zone of evanescent $P$-waves will have little effect on Rayleigh wave dispersion.

3.2.1 Propagating $P$-waves at the surface ($p < a_0^{-1}$)

Once we allow for the possibility of interfaces in our earth model we can no longer use simple asymptotic representation of the reflection coefficients from the stratification. How-
ever, we can represent them via their amplitude and phase
\[ R_D^{PP} = |R_D^{PP}| \exp(i\phi), \quad R_D^{SS} = |R_D^{SS}| \exp(i\psi), \]

where the phases \( \phi, \psi \) will differ from their asymptotic values by interface contributions. For a perfectly elastic medium we can make use of the unitarity properties of the reflection coefficients (Kennett, Kerry & Woodhouse 1978)
\[ |R_D^{PP}| = |R_D^{SS}|, \quad |R_D^{PS}|^2 = 1 - |R_D^{SS}|^2, \]
\[ \arg R_D^{PS} = \pi/2 + \frac{1}{2}(\phi + \psi), \] (3.22)
to recast (3.3) into a form reminiscent of (3.14). The secular equation becomes (Kennett 1982)
\[ \cos \frac{1}{2}(\phi + \psi) + |\tilde{R}_S^{SS}||R_D^{SS}| \cos \frac{1}{2}(\phi - \psi) = |\tilde{R}_P^{PS}||R_D^{PS}|, \] (3.23)
in which the dominant behaviour will once again arise from the phase of \( R_D^{SS} \). When we have \( P \) to \( S \) conversion within the stratification we have a double 'solotone' effect. The major contribution arises from the linking of \( P \)- and \( S \)-wave effects at the surface and the secondary contribution arises from \( P \) to \( S \) conversion within the stratification. When we allow for the latter effect the \( S \) reflection sequence associated with the presence of free surface (3.16) is modified to
\[ A_S = \tilde{R}_S^{SS} + \frac{\tilde{R}_D^{SP}R_D^{PP}\tilde{R}_D^{PS}}{1 - \tilde{R}_D^{PP}R_D^{PP}} + \frac{R_D^{SP}\tilde{R}_D^{PS}}{1 - \tilde{R}_D^{SP}\tilde{R}_D^{PS}}, \] (3.24)

Since we now have the additional possibility of alternate \( P \) to \( S \) conversion at the surface and in the stratification. The modulation of the spacing of the Rayleigh model branches in frequency depends mostly on \( [1 - \tilde{R}_D^{PP}R_D^{PP}]^{-1} \) and to a lesser extent on \( [1 - \tilde{R}_D^{SP}\tilde{R}_D^{PS}]^{-1} \). This leads to 'ghost' dispersion branches controlled by the near surface \( P \)-wave distribution which are visible only because of a change in the spacing of the \( S \)-wave dominated true dispersion curves. This effect is well illustrated in Fig. 1 for computations of Rayleigh mode dispersion for the upper mantle model T7 (Burdick & Helmberger 1978).

Figure 1. Comparison of the 'ghost' dispersion branches, due to \( P \) effects, formed by modulation of the spacing of the Rayleigh mode branches, with the \( S \)-dominated branches at large slowness, for the upper mantle model T7.
3.2.2 P-waves evanescent at the surface \((a_0^{-1} < p < \beta_0^{-1})\)

Once P-waves become evanescent throughout the stratification, the propagation pattern simplifies considerably even in the case of general stratification. P-waves will only be significant near the surface. As a result the ratio \(y_o = s(0)/(\rho_0 \beta_0^2 u(0))\) will be very close to its value in a smoothly varying medium, unless there is a strong interface close to the surface.

The SV-wave contribution to the secular equation \(\gamma_p\) will be affected by the presence of the structure in the stratification. At any interface there will be some coupling of the SV variables \((w, t)\) into \((u, s)\), but since P-waves are evanescent the \((u, s)\) contribution will decay away from the interface at a rate that increases rapidly with frequency.

With a solution for \(w, t\) which decays as \(z \to \infty\), the secular function for Rayleigh waves has the asymptotic form

\[
y_p \sim (\xi^2 |q_o|)^{-1}.
\]

The right-hand side of (3.25) is slowness-dependent but not frequency-dependent. This approximation corresponds to neglecting all terms in \(R_D^{PP}\) and \(R_D^{PS}\) in (3.3) so that we require

\[
R_D^{SS}(\omega, p) \tilde{R}^{SS}(\omega, p) \sim 1.
\]

The dispersion is therefore almost entirely controlled by the shear wave distribution and (3.26) differs from the corresponding result for Love waves, \(R_D^{HH} = 1\), by the slowness-dependent term \(\tilde{R}^{SS}\). For perfectly elastic medium both \(R_D^{SS}\) and \(R_D^{PS}\) are unimodular and the full dispersion relation, including P-dependent terms, can be written in the form (Kennett 1982)

\[
\sin \left(\frac{\psi + \psi_0}{2}\right) = \left|\tilde{R}^{PS}\right| \left[2|\tilde{R}_D^{PP}| \sin \phi\right]^{1/2} - \left|\tilde{R}_D^{PP}\right| \sin \left(\frac{\phi + \psi_0 - \psi}{2}\right),
\]

where \(\psi_0\) is the phase of \(\tilde{R}^{SS}(p)\). The rhs of (3.27) diminishes rapidly with increasing frequency and for even moderate frequencies can be thought of as a small frequency perturbation to the main dispersion behaviour given by the vanishing of the lhs. This result has been used as the basis of a rapid method of calculating Rayleigh wave dispersion by Kennett & Clarke (1983)

In terms of the SV variables \(w, t\) the approximate dispersion relation (3.25) can be viewed as imposing a slowness-dependent surface condition for Rayleigh waves as opposed to the zero traction condition for Love waves. At fixed slowness we can then think of \(\beta w\) as an SV eigenfunction of the Rayleigh wave system. We introduce the factor of \(\beta\) as in Section 3.1.2 to give the closest possible correspondence with the Love wave results.

If this identification is correct we would expect the shape of \(\beta w\) to be similar to that for the corresponding Love mode. In Fig. 2 we compare the SH eigenfunction for modes 1–9 with \(\beta w\) and the P contribution \(u\) calculated from the eigendisplacements \(U, V\) for the corresponding Rayleigh modes. We note that as expected the variable \(u\) only differs from zero near the surface and at interfaces and the rate of decay in this contribution increases with mode number as the frequency increase. The general character of the SV and SH functions are similar and for the highest mode the curves almost overlay. Since \(\beta w\) is not continuous at an interface there are slight jumps in the curves which do not affect the shape of the eigenfunction significantly.

We can therefore regard \(\beta w\) as an approximate eigenfunction for Rayleigh waves in the large slowness regime. We note that the order of the mode corresponds to the number of zero-crossings of the auxiliary variable \(\beta w\), an analogous result for the well-known
property of the Love wave eigendisplacement. The orthogonality relation for different Rayleigh modes at fixed slowness is of rather daunting form and involves all the displacements and stresses \( U, V, P, S \) (cf. Kennett 1981). However, numerically, at least, we have a simple property in terms of \{\beta w\}. Writing \( w^n \) for the eigenfunction in the \( n \)th mode we find

\[
\int_0^\infty dz \beta^2 \{\beta w^n(z)\} \{\beta w^m(z)\} \sim \delta_{nm}
\quad (3.28)
\]

with a very small error associated with very weak coupling into modes \( n - 1, n + 1 \). It is not surprising that we need no \( P \) contribution to (3.28) since \( P \) is only non-zero over very limited regions (cf. Fig. 2). This approximate orthogonality relation greatly simplifies the use of modal expansions to describe Rayleigh waves in laterally varying media (Clarke 1982).

4 Discussion

In this paper we have shown how the separation of the seismic wavefield into \( P \)- and \( SV \)-wave contributions can be used as an effective tool for the understanding of Rayleigh wave dispersion. The principal coupling of \( P \)- and \( SV \)-waves occurs through the free-surface boundary conditions and when this is the dominant contribution the secular equation reduces to a product of terms associated with independent \( P \)- and \( SV \)-wave propagation, plus a surface term. When significant \( P \) to \( S \) conversion occurs within the stratification the position is rather more complex. However, when \( P \)-waves are evanescent throughout the half-space it is possible to work in terms of an \( SV \) variable and find an approximate scalar eigenfunction \{\beta w\} for the Rayleigh wave system. The major contributions to \( w \) are governed by a second-order differential equation rather than the full fourth-order Rayleigh system. The relation of this approach to the method proposed by Keilis-Borok, Neigauz &
Shkadinskaya (1965) for calculating Rayleigh wave dispersion is intriguing but has not yet been fully elucidated.

Although our entire explicit treatment has been for isotropic media, the results could be extended to transversely isotropic media. The transformation to $u, s, w, t$ would depend on the full set of elastic constants $A, F, C, L, N, p$. The dispersion properties of Love and Rayleigh waves would now be more different because the vertical phase integrals such as $\int q_\theta dz$ would have different vertical slownesses for $SH$- and $SV$-waves. However, the entire set of results in terms of reflection coefficients are still valid and so could be applied to the dispersion problems discussed by Anderson & Dziewonski (1982).

In this paper we have concentrated on the major behaviour of surface wave dispersion in a stratified medium. Superimposed on this can be additional complications associated with the presence of velocity inversions within the stratification. The resulting character of the dispersion curves and their relation to the reflection properties of the stratification for this case are discussed by Kerry (1981a, b).

References
