AN OPERATOR APPROACH TO FORWARD MODELING, DATA PROCESSING AND MIGRATION*

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ABSTRACT

The response of a seismic model to excitation by a source can be represented in terms of the action of reflection and transmission operators for portions of the structure. This approach provides a flexible framework for both modeling and processing problems.

The operator development provides a physical description of the wave propagation process and, via the expansion of reverberation operators, gives a mechanism for assessing the accuracy of approximate developments. The representation suggests new ways of developing modeling algorithms by balancing the computational effort expended on minor and major features of the model.

For processing problems, the operator representation shows the relation of processing stages to the seismic wave field and thereby indicates effective sequences of operations. For migration it is possible to specify an ideal pre-stack migration procedure in terms of the inverse of the propagation operators and to examine the problems which need to be overcome by practical algorithms.

INTRODUCTION
For complex geological models the propagation of seismic waves from source to receiver is very difficult to describe in simple terms. A number of methods have been developed to allow seismic modeling without excessive computational effort, but these depend on either approximations, such as the neglect of multiples in ray-based methods, or on simplifications of the model, as in finite difference and finite element techniques.

The object of this paper is to present a simple, physically-based formalism for seismic modeling which, in principle, allows a complete specification of the response

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of a model to excitation by a source. We present a scheme based on the cumulative action of reflection and transmission operators for portions of the medium which allows the sequence of propagation processes to be followed. This approach enables the approximations inherent in economical modeling schemes to be clearly identified and also provides a framework which can help with the interpretation of the results of purely numerical methods. The operator development may be used for both elastic and acoustic waves, and for solid models can provide a systematic means of following interconversions between P- and S-waves. In a number of cases explicit representations may be found for the reflection and transmission operators, and this suggests the development of new modeling techniques.

With the aid of our operator representation of the medium response we are able to examine the action of commonly used seismic processing operators as they interact with the seismic wave field. A particularly important class of such operators arises in migration and here we are able to specify an ideal prestack migration scheme and also to investigate the extent to which feasible schemes can give an adequate representation of the ideal.

Because we are dealing with the cumulative action of physical propagation processes, the order in which the reflection and transmission operators act is very important. When we attempt to unravel the seismic wave field in order to determine the structural model, we must also be careful about the order in which the processing operations are applied. The use of the operator formalism provides a clear specification of the optimum order in a processing sequence.

REFLECTION AND TRANSMISSION OPERATORS

As a preliminary to investigating the seismic response of an entire model we introduce the concept of reflection and transmission operators which are the principal building blocks in our development.

A flat interface

For simplicity we start with a plane horizontal interface at the level $z_I$ lying between two uniform media. A given incident downward wave $D^i(x, t)$ upon the interface will generate an upward reflected wave $U^r(x, t)$ in the same medium. If we make a plane wave decomposition of these two wave fields, they are related by the action of the reflection coefficients at the interface $z_I$ (e.g., see Kennett 1983). Working in terms of frequency $\omega$ and horizontal slowness $p = (p_x, p_y)$, the downward field at the interface $z_I$ may be split into plane waves as

$$D(x_\perp, z_I, \omega) = \frac{1}{2\pi} \int \int dp_x dp_y \exp (i\omega p \cdot x_\perp) \omega^2 \mathcal{D}(p, z_I, \omega),$$

where $x_\perp = (x, y)$ is the position vector in the horizontal plane. The reflected wave field at $z_I$ may then be built up from the slowness components $\mathcal{D}(p, z_I, \omega)$ by the
action of the interfacial reflection coefficients $R'_D$ for a downward travelling incident wave. Thus,

$$U(x_\perp, z_I, \omega) = \frac{1}{2\pi} \int \int dp_x \, dp_y \exp\left(i\omega p \cdot x_\perp\right) \omega^2 R'_D(p) D(p, z_I, \omega). \quad (1)$$

$R'_D(p)$ includes the reflection coefficients for slowness $p$ and is independent of frequency. For acoustic waves $R'_D$ is just a scalar, but for elastic problems $R'_D$ would be a $3 \times 3$ matrix of coefficients. In the case of isotropic media $R'_D$ factors into a $2 \times 2$ matrix coupling $P$- and vertically polarized $S$-waves (SV) and a scalar coefficient for horizontally polarized $S$-waves (SH). For each slowness the reflected field is obtained by multiplying the component of the incident field by the appropriate reflection coefficient.

When the spatial response (1) is reconstructed via a Fourier integral over horizontal wave number, we recognize that there is an alternative representation as a spatial convolution:

$$U(x_\perp, z_I, \omega) = \omega^2 \tilde{R}'_D(\omega x_\perp) D(x_\perp, z_I, \omega), \quad (2)$$

where

$$\tilde{R}'_D(x_\perp) = \frac{1}{2\pi} \int \int d^2 p R'_D(p) \exp\left(i p \cdot x_\perp\right)$$

is the inverse transform of the reflection coefficients with respect to the horizontal slowness. The explicit frequency dependence in (2) arises from the fact that $R'_D(p)$ is independent of frequency.

Since the expansion (2) must allow for all upward travelling waves generated at the interface, $\tilde{R}'_D(x_\perp)$ will include the action of both pre- and postcritical reflections and head waves. Although the main concentration of the spatial reflection term $\tilde{R}'_D(x_\perp)$ will therefore be near the origin, there will be a long tail to include the decay of head waves with distance. We illustrate this behavior in fig. 1 for the $P$-wave reflection coefficient from an interface between two elastic media. In fig. 1(a) we display the slowness dependence of the amplitude and phase of the reflection coefficient. Since the inversion integral is over all slowness we have to extend the definition of the reflection coefficient into the zone where $P$-waves are evanescent in the upper medium. The onset of evanescence is marked by a star. In fig. 1(b), we show the result of the inverse transformation over slowness to construct $\tilde{R}'_D(\omega x)$. We have plotted both the real and imaginary parts of this complex function. The real part describes the reflection in phase with the incident wave which is of greatest interest in reflection work and the imaginary part of the reflection with a 90° phase shift, arising from wide angle reflections and head waves. As expected, $\tilde{R}'_{pp}(\omega x)$ has its greatest values near the origin and after an initial rapid drop there is an abrupt change in the gradient of the amplitude around $\omega x = 1000$ m/s; for larger arguments the amplitude decays fairly slowly. The structure in this tail reflects the details of the slowness behavior illustrated in fig. 1(a). The oscillatory behavior of these spatial reflection terms, as we move away from the geometrical reflection point, is in
Fig. 1. (a) The phase and amplitude behavior of the PP-reflection coefficient between two elastic media as a function of slowness $p$. The P-velocities: $\alpha_1 = 2000$ m/s and $\alpha_2 = 3000$ m/s; S-velocities: $\beta_1 = 1000$ m/s and $\beta_2 = 1600$ m/s, densities $\rho_1 = 2.3$ Mg/m$^3$ and $\rho_2 = 2.5$ Mg/m$^3$, loss factors $Q_\alpha^{-1} = 0.01$ and $Q_\beta^{-1} = 0.02$ in both media. (b) Spatial behavior of the corresponding reflection operator.
accord with the Fresnel zone picture of the reflection process (Born and Wolf 1959, chapter 8).

In a ray treatment we would assume a pointwise operator acting just at the origin, but we see that for any finite frequency a larger region is involved in reflection. In our illustrated example the principal contribution comes from a zone about 2000/\omega m wide for angular frequency \omega, i.e., about a sixth of a wavelength across. A merit of the representation (2) is that it shows how the effective size of the zone which contributes to the reflection scales inversely with frequency.

Let us now introduce the \textit{reflection operator} \textbf{R}^I_0 for this interface, defined such that \( \textbf{U}^o \) is to be found by the action of \( \textbf{R}^I_0 \) on the incident field:

\[
\textbf{U}(x_\perp, z_I, \omega) = \textbf{R}^I_0[\textbf{D}^o(x_\perp, z_I, \omega)].
\]

For this planar interface comparison of (2) and (3) shows that we must identify \( \textbf{R}^I_0 \) with the spatial convolution operation \( \omega^2 \tilde{R}_0^I(\omega x_\perp)^* \). In a similar way we can introduce the corresponding transmission operator \( \textbf{T}^I_0 \) for downward incident waves as a spatial convolution with the inverse Fourier transform of the plane wave transmission coefficients as a function of horizontal position.

Once the interface deviates from a plane we lose the spatial convolution properties, although for small, slow perturbations (2) will often provide an adequate approximation.

\textbf{A non-planar interface}

For reflection from a non-planar interface \( h_I(x) \), we can extend the idea of a reflection operator \( \textbf{R}^I_0 \) connecting the ingoing and outgoing fields at the interface. We consider waves emerging from a point source \( x_s \) and received at a point \( x_R \). In terms of a generic point \( \xi \) on the interface, the \textit{ingoing} field will be a linear combination of the Green's tensor \( G^o(\xi, x_s) \) for the region \( a \) containing \( x_s \) and \( x_R \), and its spatial derivatives. The outgoing (reflected) field will depend on \( G^o(x_R, \xi) \). Following the treatment of Kennett (1984a) we can look for the wave field at the receiver in the form

\[
\textbf{V}(x_R) = \textbf{D}^o(x_R, x_s) + \int_I d^2\xi \textbf{G}^o(x_R, \xi) \cdot \textbf{R}_D(\xi; \textbf{D}^o),
\]

where the integral is to be taken over the interface \( I: h_I(x) \), and \( \textbf{D}^o \) is the field which would be present in the absence of the interface. This representation is equivalent to regarding the reflected waves from the interface as generated by a secondary force system \( \textbf{R}_D(\xi; \textbf{D}^o) \) distributed along the interface. The force terms \( \textbf{R}_D \) can be found as the solution of an integral equation in terms of the incident field \( \textbf{D}^o \) and the Green's tensors for the two media abutting the interface:

\[
\textbf{L}(x_R; \textbf{D}^o) = \int_I d^2\xi \textbf{K}(x_R, \xi) \cdot \textbf{R}_D(\xi; \textbf{D}^o).
\]
The integral kernel $K$ depends on the difference in the Green's tensor for the two sides $(a, b)$ of the irregular interface $I$:

$$K_{kp}(x_R, \xi) = \int_I d^2\eta [G_{ka}(x_R, \eta)H^a_{qp}(\eta, \xi) - H^b_{ka}(x_R, \eta)G^a_{qp}(\eta, \xi)]$$

where $H$ is the traction on $I$ associated with the Green's tensor $G$, and the integration is taken over the whole surface. The dependence on the incident field appears in $L$:

$$L_k(x_R; D^a) = \int_I d^2\eta [G_{ka}(x_R, \eta)E^a_{pq}(\eta, x_s) - H^b_{ka}(x_R, \eta)D^a_{pq}(\eta, x_s)]$$

where $E^a$ is the traction produced by the incident wave. This operation removes any part of the incident field which resembles the transmitted field in medium $b$ and so reveals the reflected contribution. The full mathematical development is to be found in Kennett (1984a). In general, the solution of (5) has to be sought numerically via a discretization of the interface and the reduction of the integral equation (5) to a set of linear equations. The kernel $K$ depends only on the properties of the media adjoining the interface and so many different incident fields can be considered without the need for recomputing the inverse of $K$. In the special case when the interface is flat, the solution can be found by transform methods and we recover the representation (2).

Since we are now able to find the force terms $R_D$, equation (4) enables us to determine the upward field at the receiver in terms of the downward field $D^a$ at the interface. This enables us to identify the action of the reflection operator for the interface as equivalent to the integral term in (4). Thus, we can write

$$U^a(x_R) = R^a_{D}(D^a)$$

$$= \int_I d^2\xi G^a_{ka}(x_R, \xi)R_D(\xi; D^a)$$

and, as for the flat case, the full reflective effect is distributed over the interface. In a ray treatment we would have a point operator at the geometrical reflection point. For all finite frequencies the full representation (6) gives a concentration near this point but allows for the possibility of diffraction effects. As in the flat case, there will be a decaying oscillatory pattern in the force distribution as $\xi$ moves away from the reflection point.

Effective approximate results for the reflection from an irregular surface can be obtained by making simple approximations for the form of the incident wave and the Green's functions for the media on the two sides of the interface, but retaining the full surface integral. This will give an approximate integral equation for the secondary forces $R_D$ but the reflected field will be well represented. In particular for a truncated reflector, such an approach will give a much better treatment for the diffracted energy than the usual approach of scaling the results for a rigid screen (e.g., see Trorey 1970). Such an approach allows for the angle dependence of the reflection from an interface between two dissimilar media, and for elastic models
both P- and S-wave diffraction to be modeled at the same time. The method may be regarded as an enhancement of the current Kirchhoff approximations for interface effects.

In transmission through an interface we take a similar representation of the reflected field to (6), by working with a secondary force system $T_D$ along the interface. The integral over the surface with the Green’s function for the lower medium can then be identified with the action of the transmission operator for the interface; so that the transmitted field

$$D^b(x_T) = T'_D(D^a),$$

$$= \int d^2\xi G^b(x_T, \xi) T_D(\xi; D^a).$$  \hspace{1cm} (7)

*The effect of a region*

For a single interface we have established a means of generating the reflected and transmitted fields for a given incident wave. The method we have used—with a representation by secondary force systems—can be extended to regions of a model (Kennett 1984a). For weak perturbations in material properties about a horizontally stratified medium, an alternative approach may be used. With a Born series development for the scattered field from the inhomogeneities (e.g., see Hudson 1968, Clayton and Stolt 1981) we get a representation in terms of a volume distribution of source terms. To first order these terms depend on the wave field in the background medium and are linear in the size of the inhomogeneity. The reflection and transmission operators can then be built up by combining the weak scattering from the heterogeneity with the major effects of the horizontal stratification.

How then do we build up the response of a composite region, if we know the reflection and transmission operators for its constituent parts? The answer lies in following the cumulative propagation processes and may be illustrated by reference to fig. 2.

For a region bounded above by an interface $A$, below by an interface $C$ and divided into two parts by an interface $B$, the action of the overall reflection operator on an incident downward field may be represented as

$$R^{AC}_D(D^a) = R^{AB}_D(D^a) + T^{AB}_U R^{BC}_D T^{AB}_D(D^a) + T^{AB}_U R^{BC}_D R^{AB}_U R^{BC}_D T^{AB}_D(D^a) + \ldots,$$  \hspace{1cm} (8)

where each operator acts on the quantity to its right. Thus the overall reflection is composed partly of reflection from $AB$ represented via $R^{AB}_D$, and partly of reflection from $BC$ after transmission down and back through $AB$. The first such contribution is $T^{AB}_U R^{BC}_D T^{AB}_D$, where the transmitted field $T^{AB}_D(D^a)$ is treated as the downward field on which $R^{BC}_D$ acts. Then the upward transmission $T^{AB}_U$ acts on the field $R^{BC}_D T^{AB}_D(D^a)$ after reflection. Subsequent reflections back from $BC$ involve internal reverberations in the entire region $AC$ with both reflection back from $AB$ and $BC$ before the wave energy emerges from $AC$. The entire sequence of possible propagation effects can be represented via a single formal expression (Kennett 1984b):

$$R^{AC}_D = R^{AB}_D + T^{AB}_U R^{BC}_D (I - R^{AB}_U R^{BC}_D)^{-1} T^{AB}_D.$$  \hspace{1cm} (9)
Fig. 2. Configuration of a region $AC$ divided into two parts $AB$ and $BC$ by the surface $B$ with a schematic representation of the construction of the reflection and transmission operators for the region $AC$ from the operators for $AB$ and $BC$.

The inverse operator $(I - R_U^A B R_D^B)^{-1}$ represents the cumulative effect of the entire sequence of internal reverberations in $AC$, since we have the identity

$$ (I - R_U^A B R_D^B)^{-1} = I + R_U^A B R_D^B + R_U^A B R_D^B R_U^A B R_D^B (I - R_U^A B R_D^B)^{-1}, $$

in which we can recognize those parts of the field we have already encountered. Further application of (10) generates the complete sequence of reverberations.

The overall transmission operator $T_D^{B C}$ can also be expressed in terms of the same reverberation operator

$$ T_D^{B C} = T_D^{B C} (I - R_U^A B R_D^B)^{-1} T_D^{A B}. $$

The addition rules for reflection and transmission operators (9, 11) are identical to those previously established for reflection matrices in the frequency/slowness domain for the case of horizontally stratified media (Kennett 1974), as we would expect from our discussion of the flat interface problem. For isotropic elastic materials, we would partition the reflection operators into the parts affecting P- and S-waves so that, e.g.,

$$ R_D^{B C} = \begin{bmatrix} R_{pp}^{B C} & R_{ps}^{B C} \\ R_{sp}^{B C} & R_{ss}^{B C} \end{bmatrix}_D, $$

where $R_{ps}^{B C}$ is the operator which generates an upgoing P-wave from a downward S-wave. The full suite of P- and S-wave interactions can be followed by compound-
ing the partitioned forms for the operators. These combine as if matrices were being multiplied, so that

\[
\mathbf{R}_D^{BC} \mathbf{T}_D^{AB} = \begin{bmatrix}
\mathbf{R}_{pp}^{BC} \mathbf{T}_{pp}^{AB} + \mathbf{R}_{ps}^{BC} \mathbf{T}_{sp}^{AB} & \mathbf{R}_{ps}^{BC} \mathbf{T}_{ps}^{AB} + \mathbf{R}_{ps}^{BC} \mathbf{T}_{ss}^{AB} \\
\mathbf{R}_{sp}^{BC} \mathbf{T}_{sp}^{AB} + \mathbf{R}_{ss}^{BC} \mathbf{T}_{sp}^{AB} & \mathbf{R}_{sp}^{BC} \mathbf{T}_{ps}^{AB} + \mathbf{R}_{ss}^{BC} \mathbf{T}_{ss}^{AB}
\end{bmatrix}.
\]

Identification of multiple paths requires the expansion of the reverberation operator (10).

Now that we are able to add together the reflection and transmission effects of two regions we can extend the results to multiple regions by cascading the procedure. The addition of further regions to \( AC \) requires that the reflection and transmission operators for \( AC \) be combined with those for the new region using the relations comparable to (9) and (11).

The idea of a reflection operator may also be used for waves incident from below on a region bounded by a free surface. To distinguish such operators, we will use the notation \( \mathbf{R}_U^{FS} \) for reflection from the region between the interface \( S \) and the free surface. The closest analogue to transmission in this case is the relation of the surface disturbance to the incident field at \( S \). We can describe this by the action of a transfer operator \( \mathbf{W}_U^{FS} \) (Kennett 1984a).

Many of the matrix operators invoked by Berkhout (1983) can be regarded as special cases of the operators developed here. The abstract formalism has, however, the merit of a more compact and direct development.

**Forward Modeling**

In order to be able to make use of the reflection and transmission operator development of the seismic wave field, we have to introduce the effects of a source. Our description must take care of the attributes of the source, and the way in which it interacts with the medium in its immediate vicinity.

There are three source situations of major significance in seismic prospecting work. Firstly, a pressure source in water (typically an air-gun or water-gun array); this generates purely compressional (P) wave energy and P- to S-wave conversion can only occur beneath the sea-bed. Secondly, a buried source in a solid medium, such as an explosive charge; the principal radiation consists of P-waves although the surface "ghost" may contain significant S-wave energy generated by conversion at the free surface. The third case is a surface source on land where the main effect is a vertical force, e.g., a weight drop system or a standard vibrator. Now both P- and S-energy is generated at the surface and there is commonly strong coupling into surface waves ("ground roll"); for torque sources or horizontal vibrators most of the energy is in the form of shear waves, although again ground roll can be significant.

*The inclusion of a source*

The operator description of the modeling scheme takes into account the sequential action of the propagation processes above and below the source, as illustrated in fig.
3. We consider a model lying between the free surface and a lower surface $L$, divided by a horizontal plane through the source level $S$.

We envisage the source to be placed in a uniform medium, in which case we can make an unambiguous separation of the radiation into upward ($U^S$) and downward ($D^S$) parts, simply by relation to the plane $S$ (Kennett 1984b). We now regard the region above the source to abut the uniform medium and to be irradiated by the upgoing field $U^S$. Similarly the region below the source is taken to have an incident downward field $D^S$ from the uniform medium. We now have just the situation for which we have defined our reflection and transmission operators and so may represent the response in terms of operators for the regions $fS$ and $SL$.

We start with the energy radiated upward from the source. At surface receivers there will be a direct wave described by $W_U^{fs}(U^S)$, where $W_U^{fs}$ is the transfer operator from $S$ to the surface. In addition there will be reflection back to the source level, so that the total downgoing wave field at $S$ is composed of the source radiation $D^S$ and the surface "ghost" $R_{US}^{fs}(U^S)$. In a solid medium, the surface reflection will include conversion to S-waves, which can be quite important in wide angle reflections. The downward field $D^S + R_{US}^{fs}(U^S)$ is now incident on the lower part of the model containing the structures of interest. The once-reflected field is then represented by $R_{SL}^{fs}[D^S + R_{US}^{fs}(U^S)]$. The operator $R_{SL}^{fs}$ would normally be constructed by building up the response of various parts of the model using (9). $R_{SL}^{fs}$ will therefore contain all internal multiples in the region of the model below the source level.

At this stage, with only a single reflection back to the source level, we have an upward field at $S$ of

$$R_{sl}^{fs}[D^S + R_{US}^{fs}(U^S)]$$

and at surface receivers a disturbance of $W_U^{fs}R_{D}^{sl}[D^S + R_{US}^{fs}(U^S)]$. However, we still have the possibility of surface generated multiples. Each successive reflection above and below the source level requires the application of the joint operator $R_{D}^{sl}R_{US}^{fs}$ to the previous wave field.
For the seismic disturbance \( w_0 \) at the surface (e.g., the recorded particle velocity), this gives a propagation sequence including free-surface multiples of the form

\[
\begin{align*}
  w_0 &= W^{fs}_U(U^S) + W^{fs}_U(1 + R^S_D R^S_U + R^S_D R^{S S}_D R^S_D + \ldots) R^S_D [D^S + R^{fs}_U(U^S)],
\end{align*}
\]

(15)

which by analogy with (9) we can express formally as

\[
\begin{align*}
  w_0 &= W^{fs}_U(U^S) + W^{fs}_U(I - R^S_D R^S_U)^{-1} R^S_D [D^S + R^{fs}_U(U^S)].
\end{align*}
\]

(16)

If we are working with pressure receivers at the source-level, we would replace \( W; \) by an operator \( W(I + R^S) \) generating pressure from the particle velocity field in the fluid with allowance for near receiver surface reflection. For a surface source, we can envisage the level \( S \) brought right up to the surface so that \([D^S + R^{fs}_U(U^S)]\) will just be the net downward radiation.

**Separating shallow propagation**

Although (16) gives a complete description of the surface disturbance due to the source it still does not meet all of our needs. In many practical situations the source lies in a zone of relatively low seismic wave velocities underlain by material with higher wave velocities. This is the normal case in marine surveys and frequently occurs with the weathered zone on land. Since the principal interest in seismic prospecting is in energy returned from depth, we would like to separate the effects of propagation in the shallow wave guide from the energy return from depth.

We have already noted the correspondence between the results of the reflection operator method and previous work with reflection and transmission matrices in transform space. We can now turn this to account by taking over results already established by Kennett (1983, chapter 9) for propagation in a medium with a surface wave guide.

If we divide the model by an interface \( J \) which lies below all the low-velocity material, we can rewrite the expression for the surface disturbance in the form (fig. 4)

\[
\begin{align*}
  w_0 &= W^{fs}_U(U^S) + W^{fs}_U(I - R^S_D R^S_U)^{-1} R^S_D [D^S + R^{fs}_U(U^S)] + W^{ij}_U(I - R^{ij}_D R^{ij}_U)^{-1} R^{ij}_D D^i.
\end{align*}
\]

(17)
The first two terms in (17) represent propagation confined to the region above the separation level \( J \) (note that they just have the form of (16) with \( L \) replaced by \( J \)). These shallow propagating disturbances will commonly have fairly horizontal propagation paths when recorded at a seismic array. Such arrivals are "ground-roll", or water-guided waves in marine work, as well as wide angle reflections from shallow reflectors.

All reflections from beneath \( J \) are included in the last term in (17), in which \( D' \) represents the net downward energy reaching \( J \). Both \( W_U^J \) and \( D' \) can be represented explicitly in terms of reverberations occurring within the zone between \( J \) and the surface:

\[
W_U^J = W_U^S (I - R_D^S R_U^S)^{-1} T_U^S
\]

with transmission up to the source level followed by multiples near the receiver and

\[
D' = T_D^S (I - R_U^S R_D^S)^{-1} E^S,
\]

which describes the action of near source reverberations on the effective source \( E^S = [D^S + R_D^S (U^S)] \) including free surface effects. The remaining part of (17), i.e.,

\[
(I - R_D^S R_U^S)^{-1} R_D^H
\]

provides a description of reflections from below \( J \), including internal multiples, and long-lag multiples involving paths with more than one reflection from beneath \( J \).

Although the expression (17) is valid for any separation surface \( J \) lying beneath the source, it is advantageous to choose it to achieve the maximum separation between the different classes of propagation. For marine models, it is therefore desirable to take \( J \) a little below the sea-bed itself so that sea-bed interactions are included in the shallow terms, even though this means that \( R_U^J \) itself will include contributions at the sea-bed as well as from the sea-surface.

An additional merit of the representation (17) as a basis for seismic modeling is that different modeling techniques are appropriate for the shallow and deep parts of the model. It may be possible, for instance, to make use of a simple horizontally-layered model of the near surface and then compute the guided propagation via integral transform methods—such an approach has been used by Kennett and Harding (1984) in a study of low-frequency guided waves in shallow water. If there are mild horizontal variations in structure, position dependent wave field continuation can be used to simulate the multiple pattern (Riley and Claerbout 1976).

For the energy returned from depth the most promising approach for economic and accurate modeling is a hybrid method using ray tracing between major interfaces, approximate calculations of reflection and transmission operators for the interfaces via (6) with approximate Green's functions, and perturbation methods for the finer scale structure. Such an approach would represent the major amplitude behavior faithfully, with an accurate treatment of diffractions. Supplementation with selected long-lag surface multiples would allow a reasonable compromise between expense and completeness. The philosophy is similar to that employed by Dere-gowski and Brown (1983), but would avoid sole dependence on ray methods.
The expression (17) applies to a single shot gather, but as we have already noted, the interfacial reflection operators do not require full recomputation for new incident fields and so the work needed for multiple sources can be reduced.

**DATA PROCESSING**

With the formal representation (17) for the seismic response of a model to excitation by a source, we are now in a position to investigate the way in which major data processing procedures interact with the seismic wave field. In (17) we have a separation of the field into two parts, shallow propagation

\[ w^s_0 = W^s_U(U^S) + W^s_U(I - R^S_D R^S_U)^{-1} R^S_D [D^S + R^S_U(U^S)] \]

confined to the region above the level \( J \) and the deeper reflections of principal interest

\[ w^d_0 = W^l_U(I - R^l_D R^l_U)^{-1} R^l_D D^l. \]

The effect of the use of field receiver groups will be to modify the operators \( W^s_U, W^l_U \) for actual recordings with a smearing over the horizontal span of the group.

In order to reduce the contaminating influence of the shallow terms (21) the standard approach is to mute out the beginning of the records in each shot spread. The time trajectory of the mute is usually chosen to eliminate the largest early energy, at the expense of information on the shallow velocity structure from wide angle reflections (Schulz 1982, Harding 1984). Where guided wave energy is important, as in shallow marine work or in some land operations with pronounced ground roll, it may be necessary to resort to frequency/wave number filtering (e.g., see Christie, Hughes and Kennett 1983), or more delicate filtering in intercept time/slowness space (Kennett and Harding 1984).

We can represent the action of muting and filtering by an operator \( \mathcal{M} \). In an ideal situation we would have just removed the portion (21) of the response, but the deeper reflections will not emerge unscathed. We will have

\[ \mathcal{M} w^d_0 = \mathcal{M} W^l_U(I - R^l_D R^l_U)^{-1} R^l_D D^l, \]

where, e.g., the outer parts of reflection hyperbolae may be lost. The action of \( \mathcal{M} \) cannot be represented as just a modification of the reflection \( R^l_D \), since it does not commute with the other operators in (23); reverberation sequences are also modified.

The expressions (21) and (22) represent the seismic field at surface receivers for a single shot, but can be regarded as one of an ensemble of realizations for different shots. For a single shot gather, data processing operators will therefore act from the left on \( w_0 \), as for muting in (23). But operations on single receiver gathers involve a suite of shots and must act on the source terms. We can represent the action of such processes by an operator acting from the right as \( w_0 \mathcal{M} \). Unfortunately, there is no easy way to represent the conventional common mid-point gathers since this mixes combinations of many shot-receiver pairs, and has no direct relation to the propaga-
tion pattern, except for horizontally stratified media where there is no distinction between the gathers.

Although some processing steps can be applied at any stage, in many cases the optimum effect requires that a specific sequence be followed. For example, let us consider the action of wavelet extraction and stacking. With a single source gather we can attempt to extract a source wavelet by deconvolution because, if array directivity is not too strong, we will have a similar source time function on each trace. This deconvolution operator will commute with the muting process and the reflection operators (if it is one-sided), so that it has only to act on the effective source radiation $D^j$:

$$\mathcal{D}\mathcal{M}w_0^{de} = \mathcal{M}W_{U}^{ij}(I - R_{D}^{II}R_{U}^{ij})^{-1}R_{D}^{II}\mathcal{D}(D^j).$$

(24)

If we now perform an NMO stack across this gather, the effect will be to enhance the primary reflections from below $J$ and to reduce surface multiples. We can expand the reverberation sequence to emphasize the primary return by using the identity

$$(I - R_{D}^{II}R_{U}^{ij})^{-1}R_{D}^{II} = R_{D}^{II} + (I - R_{D}^{II}R_{U}^{ij})^{-1}R_{D}^{II}R_{U}^{ij}R_{D}^{II}.$$ 

With a stacking operator we can now write the result of stacking after a wavelet extraction in a form which separates the surface multiples:

$$\mathcal{D}\mathcal{M}w_0^{de} = \mathcal{M}W_{U}^{ij}\mathcal{I}R_{D}^{II}\mathcal{D}(D^j) + \mathcal{M}W_{U}^{ij}\mathcal{I}(I - R_{D}^{II}R_{U}^{ij})^{-1}R_{D}^{II}R_{U}^{ij}R_{D}^{II}\mathcal{D}(D^j)$$

(25)

and in optimum circumstances $\mathcal{I}R_{D}^{II}$ will largely remove the reverberation operator in the second term, and $\mathcal{I}R_{D}^{II}$ will be close to $R_{D}^{II}$ itself.

When wavelet extraction is applied after stacking so that we work with $\mathcal{D}\mathcal{I}\mathcal{M}w_0$ rather than (25), the multiple suppression is not affected. But now, the stretching of the outer traces in the gather under NMO correction during stacking distorts the spectrum and after summation there will be no consistent wavelet for the whole trace. For common mid-point gathers the situation is compounded by including different shots and receiver response in the stacking process.

In practice, of course, it may be necessary to resort to post-stack wavelet extraction because the statistical improvement, when traces are noisy, outweighs the distortions produced in the result. Similarly prestack equalization procedures, e.g., for land sources, can repair many of the problems with different shots and receivers.

**Migration**

The aim of a migration procedure is to attempt to reverse the pattern of the propagation effects which generated the recorded seismograms, with the object of building an image of the subsurface structure. The accuracy of reconstruction depends on the approximations made in particular algorithms and the extent to which the velocity distribution in the model is known.

How then does migration fit into the general operator scheme we have discussed in this paper? The key to this lies in reorganizing the assumption which is built into
most migration schemes that previous processing has left the ideal case of a reflection operator $R^I_D$ with only primary reflections included, acting on a broad band source signal with no ghosts. Ambient noise and residual multiples will contaminate the actual solution, but let us see what is needed in principle.

We discuss an ideal pre-stack migration procedure which shows the way in which possible processing steps can be related to the physical character of the wave propagation. We start from the expression (22) for the deep reflections and assume that the purely shallowly propagating waves have been removed. Reinstating the full forms of $W^I_U$ and $D^I$, we can write (22) as

$$w^{de}_0 = W^{fs}_U Z_R (I - R^I_D R^I_U)^{-1} R^I_D Z_S E^S,$$

where from (18) and (19), $Z_R$ and $Z_S$ are operators including reverberations in the shallow structure, near the receiver and source respectively. We recall that

$$E^S = D^S + R^S_I(U^S),$$

the effective radiation from our shallow source including surface reflections.

With an ensemble of sources and receivers we want to get at the reflections from depth and so we want to disentangle the near-receiver and near-source effects. By working with a set of records from a common source we can try to remove receiver coupling effects and the shallow reverberations: formally, this requires the action of $Z_R^{-1}(W^{fs}_U)^{-1}$ on $w^{de}_0$ from the left. Now, reorganizing into common-receiver-gathers, we endeavor to deconvolve the wavelet and to remove source directivity and the near-source shallow reverberations. As discussed above, this is represented by the action of $(E^S)^{-1}Z_S^{-1}$ from the right.

At this stage then we have

$$Y = Z_R^{-1}[(W^{fs}_U)^{-1}w^{de}_0(E^S)^{-1}]Z_S^{-1} = (I - R^I_D R^I_U)^{-1} R^I_D$$

and the operators at the right are to be envisaged as acting on an ideal isotropic delta-function source. For solid media, as pointed out by Kennett (1979), the extraction of the receiver and source directional effects contained in the central bracket in (28), requires three-component information and multiple sources at the same location. This is needed to generate sufficient information to construct the inverse matrix operators which are needed to allow simultaneous treatment of the P- and S-wave fields. Normally, the S-wave contribution to vertical component sensors is ignored (even though it is responsible for ground-roll), and an ad-hoc scalar treatment is used. However, if shear effects are neglected, we cannot expect to get accurate representations of transmission operators and their inverses. For sources in water the operators are scalar and there is no need for multiple experiments. In any practical situation, we would have an imperfect knowledge of $y$ since the dereverberation, at least, will be incomplete with finite amounts of data.

However, once we are given $y$ we can follow the prescription suggested by Kennett (1979) and recently elaborated by Berkhout (1983) and aim to remove the long-lag surface multiples in a single operation. If we have an adequate knowledge of the model to construct an accurate estimate of $R^I_U$, we can extract $R^I_D$ as

$$R^I_D = y(I + R^I_U Y)^{-1}.$$
We would normally need to expand the operator inverse resulting in a sequence of terms with alternating sign; with imperfect knowledge of y it can be difficult to control the stability of the free-surface multiple removal.

Nevertheless the estimate of the action of $R^L_D$ obtained from (29) is the starting point for migration. Let us start at the top of the model and assume that we have sufficient information about the large-scale structure of the model to enable us to find any requisite reflection and transmission operators. Consider a small slice of the reflection response by splitting at a surface $K$ corresponding to a small time step down the seismograms. We can split up the operator $R^L_D$ to expose the properties of the slice $JK$ by using (9):

$$R^L_D - R^K_D = T^K_U (I - R^K_D R^K_U)^{-1} R^K_D T^K_D.$$  

For a small enough slice $R^K_D$ will be dominated by primaries and can be directly identified. The reflection operator for the zone below $R^K_D$ can be found by downward continuation and multiple removal. We construct

$$Q = (T^K_U)^{-1} (R^L_D - R^K_D) (T^K_D)^{-1}$$

by downward continuation on both source and receiver gathers—the so called "double square root" procedure applies for a uniform medium (e.g., see Berkhout 1983). We then have to extract the action of $R^L_D$ from $Q$ as

$$R^L_D = Q (I + R^K_D Q)^{-1},$$

with sufficient knowledge about upward propagation in the region $JK$. Once again errors will cascade to produce instability in the estimate of $R^L_D$ so that the base wave velocity distribution needs to be known well.

Once one slice has been extracted, the whole process can be repeated to give a reflection image with correctly positioned reflectors. This ideal pre-stack migration procedure depends on the accuracy of the downward continuation operators. Errors tend to build up as one attempts to move deeper in the section. In principle, if the density is known and the Born approximation is adequate the amplitudes in $R^L_D$ could be interpreted in terms of velocity variations, so that an inversion is achieved at the same time. A rather different approach taken by Lailly (1983) working directly from the equations of motion, reaches a comparable result (also using operator methods).

In practice, of course, we have limited information and thus cannot realize the ideal scheme we have just outlined, especially with regard to full inversion. However, the operator approach has the merit of laying bare the bones of the technique and revealing the most likely sources of error.

Practical migration schemes currently operate on stacked seismic data so that it is more difficult to establish a direct relation to the physical propagation processes. However, once again we can examine the way in which processing and propagation interrelate by working with the operators approach. Although we have based our discussion on surface seismic methods, similar ideas can be applied to Vertical Seismic Profiling, and operator methods provide a convenient link between the two approaches.
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