On the use of truncated modal expansions in laterally varying media

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Accepted 1987 April 28. Received 1987 March 27; in original form 1986 November 19

Summary. Coupling between modes cannot be neglected for surface-wave trains propagating through laterally varying media where the spatial wavelength of the heterogeneity is comparable to the wavelengths of the surface waves, unless the heterogeneity is very weak. For an extended zone of significant heterogeneity, first-order scattering approximations are inadequate and instead a modal evolution scheme needs to be used. The displacement and traction fields are represented by a superposition of modal eigenfunctions of a reference model, with modal coefficients which vary with horizontal position due to the action of the heterogeneity. For a 2-D heterogeneity field, the behaviour of the modal coefficients can be represented via a set of first-order ordinary differential equations in the horizontal coordinate.

For practical applications, the number of modes used in the representation has to be limited, and this places restrictions on the class of heterogeneous models which can be considered. With a fixed reference medium, only a single set of modal eigenfunctions have to be determined. However, there is a restriction to spatially bounded heterogeneity and a direct physical interpretation for the modal coefficients can only be made where there is no deviation from the reference model. The basic modal representation is effective provided that the heterogeneous model has a similar character to the reference. Perturbations of major discontinuities in the model are not well handled with a limited mode set, and should be avoided entirely for fluid/solid boundaries. The limitations on the heterogeneous model can be related to the horizontal and vertical wavenumber content of the modal eigenfunction set. To a first-order approximation, the action of the heterogeneity is to introduce coupling between modes when its spatial wavenumbers match the difference between modes. When higher order effects are included the coupling zone is somewhat broadened, but the first-order results give a useful constraint on mode coupling.
In general, the number of modes used in the modal expansion should be at least two more than is required to cover the desired range of phase velocities, in order to reduce the errors in the modal coefficients introduced by truncation. When the model consists of loosely coupled waveguides, the cut in mode number should be made so as to exploit the block structure of the coupling terms, whereby modal eigenfunctions with energy concentrated in a particular waveguide are preferentially coupled.

**Key words:** modal expansions, laterally varying media surface waves, waveguides

1 Introduction

As the seismic structure of the Earth becomes better mapped we have increasing evidence that the Earth is laterally heterogeneous on a wide range of scales and spatial wavelengths. However, we have currently a rather limited set of theoretical tools with which to attempt to study the elastic wavefield in an arbitrarily heterogeneous earth.

In the particular case of surface wave propagation in laterally varying media, a number of different methods exist which are geared to different classes of heterogeneity. When the spatial wavelength of the variations in the elastic parameters are long compared with those of the surface waves, Woodhouse (1974) showed that each surface-wave mode travels independently along rays which depend on the phase velocity for that mode in the local structure. This approach has recently been extended by Yomogida & Aki (1985) to use Gaussian beam methods along the ray trajectories, whilst retaining the independence of the modes. Coupling between such 'local modes', in the acoustic case, has been considered by Odom (1986) for rather simple models. First-order Born scattering techniques can be applied for a weak heterogeneity, such that the net perturbation to the incident wavefield remains small. This condition can be satisfied for localized significant heterogeneity or for much smaller deviations in seismic parameters spread over a larger region. Snieder (1986a, b) has derived such a Born scattering representation for surface waves, with explicit allowance for inter-mode coupling.

When the spatial wavelength of the heterogeneity is of the same general order as the wavelengths of the surface waves and its amplitude is not very small, we can neither neglect coupling between modes nor assume that the first-order Born results are adequate. Kennett (1984) described a scheme for handling guided waves in such a situation, based on a superposition of modal contributions for a fixed reference structure, with modal coefficients which vary with position. The first-order differential equations describing the evolution of the modal coefficients involve cross-coupling between the various modes. To avoid excessive computation it is desirable to restrict attention to a limited number of modes. This truncation of the expansion, together with the nature of the representation, limits the range of models for which the cumulative errors in this mode-coupling scheme can be kept in reasonable bounds.

In this article we will review the development made by Kennett (1984) and point out the features in the analysis which place restrictions on the character of the heterogeneity in the case of a truncated modal expansion. The major discontinuities in the actual model and the reference model should, as far as possible, be coincident. This reduces errors associated with the representation of the traction field, especially from discontinuities in traction components. At fixed frequency, the use of a fixed number of modal functions gives a limited resolution in depth for any details in the wavefield. Thus, if the true
displacement-field generated by the presence of heterogeneity should vary more rapidly than the eigenfunction for the highest mode retained, then there will be significant error accumulated as we carry the restricted calculation through the medium. Deviation of the elastic parameters from those of the reference model on scales smaller than that for the eigenfunction of the highest mode at the same depth cannot, therefore, be adequately represented.

Kennett's representation is most simply applied to a 2-D heterogeneity field where the medium does not vary perpendicular to the direction of travel of the surface wavetrain. This restriction still allows the application of the technique to many models of interest, e.g. crustal graben structures (Kennett & Mykkeltveit 1984). However, the coupling of Rayleigh and Love modes by 3-D heterogeneity cannot be handled in such a medium (although coupling induced by anisotropy can be considered).

When the relatively restrictive conditions for strict application of a truncated modal expansion are not satisfied there will be cumulative error build up as the surface-wave train is tracked through the medium. Nevertheless the results for a limited mode set can often give a very useful indication of the character of the wavefield.

2 Modal evolution equations

We will give a brief recapitulation of the coupled mode technique introduced by Kennett (1984) with emphasis on those features of the development which may limit practical applications. We will restrict attention to a two-dimensionally heterogeneous structure and work with a fully anisotropic medium for which a very compact notation is available. The heterogeneity is assumed to be invariant in the transverse direction to the wave vector \( \mathbf{k} \), and so the direction of this vector will remain constant even in the anisotropic case where the ray direction associated with a particular mode does not lie along \( \mathbf{k} \).

We consider the heterogeneity as a deviation from the properties of a reference model. This perturbation is not necessarily small but must be bounded in space. We seek a representation of the displacement field in terms of a superposition of the modal eigenfunctions of the reference model with modal coefficients which vary with position.

We work in Cartesian coordinate system, with propagation in the \( x-z \) plane and introduce the displacement vector \( \mathbf{w} = (w_x, w_y, w_z) \). We also need to specify the traction field. In the case of a modal wavetrain propagating in the \( x \)-direction we are concerned with continuity of traction on vertical planes and so we take \( \mathbf{t}_1 = (\tau_{11}, \tau_{12}, \tau_{13}) \). In addition, the material-continuity requirements mean that we also need to be able to refer to the traction on a horizontal plane \( \mathbf{t}_3 = (\tau_{31}, \tau_{32}, \tau_{33}) \). For the 2-D situation we need a number of combinations of elastic moduli which may be represented compactly in terms of matrices \( \mathbf{C}_{ij} \) such that

\[
(C_{ij})_{kl} = c_{klji},
\]

where the \( c_{klji} \) are the anisotropic moduli.

The equations of motion and the stress-strain conditions can be cast into a form where the derivatives with respect to the propagation direction \( x \) appear only on the left hand side of the equations

\[
\frac{\partial}{\partial x} \begin{pmatrix}
\mathbf{w} \\
\mathbf{t}_1
\end{pmatrix} = \begin{pmatrix}
A_{ww} & A_{wt} \\
A_{tw} & A_{tt}
\end{pmatrix} \begin{pmatrix}
\mathbf{w} \\
\mathbf{t}_1
\end{pmatrix} - \begin{pmatrix}
0 \\
\mathbf{f}
\end{pmatrix},
\]

in the presence of a volume force contribution \( \mathbf{f} \). The differential operators \( A_{..} \) do not
depend on the horizontal derivatives of the material properties and for the 2-D case have the explicit form

\[
\begin{pmatrix}
A_{ww} & A_{wt} \\
A_{tw} & A_{tt}
\end{pmatrix} = \begin{pmatrix}
-C_{11} \frac{\partial}{\partial z} & C_{13}^2 \\
-\rho \omega^2 I - \frac{\partial}{\partial z} \{Q \frac{\partial}{\partial z} - \frac{\partial}{\partial z}[C_{31} C_{11}^2] \}
\end{pmatrix},
\]

(2)

where we have written \( \frac{\partial}{\partial z} \) for \( \partial / \partial z \) and set

\[Q = C_{33} - C_{31} C_{11}^2 C_{13},\]

(3)

The unclosed brackets in (2) indicate that the operator acts to its right. We should emphasize that we have concentrated attention on \( x \) in equation (1) which runs counter to common usage. For example, the construction of modal eigenfunctions in stratified media depends on the equivalent first-order equations but with a differential with respect to the depth variable \( z \). Across an interface \( z = \text{const} \), the \( \tau_{13} \) component of \( \mathbf{t}_4 \) will be continuous, but \( \tau_{11}, \tau_{12} \) are not required to be continuous. On the other hand \( \mathbf{t}_3 \) will be continuous and can be represented in terms of \( \mathbf{w} \), and \( \mathbf{t}_4 \) as

\[
\mathbf{t}_3 = Q \frac{\partial}{\partial z} \mathbf{w} + C_{31} C_{11}^2 \mathbf{t}_4.
\]

(4)

The equations (1)–(3) apply in an arbitrarily heterogeneous medium but by themselves provide no direct link to the surface-wave case we wish to consider.

When the heterogeneous medium does not deviate too strongly from the reference medium, we can envisage that it may be possible to find a representation in terms of the surface-wave modes of the reference structure. In this paper we consider the case of a fixed, stratified, reference medium as used in the examples of Kennett (1984) and Kennett & Mykkeltveit (1984).

At each position \( x \) we consider cutting the actual structure along a vertical plane and then weld on the reference structure to ensure continuity of displacement \( \mathbf{w} \) and traction \( \mathbf{t}_4 \). In this reference structure, now we can represent the displacement and traction at \( x \) as a superposition of modal contributions and write

\[
\begin{pmatrix}
\mathbf{w}(x) \\
\mathbf{t}_4(x)
\end{pmatrix} = \sum_r c_r(x) \exp \left( ik_r x \right) \begin{pmatrix}
\mathbf{w}^0_r(k_r, z) \\
\mathbf{t}^0_{1r}(k_r, z)
\end{pmatrix},
\]

(5)

where the \( \mathbf{w}^0_r \) are the displacement eigenfunctions for the reference medium and the horizontal tractions \( \mathbf{t}^0_{1r} \) are derived from \( \mathbf{w}^0_r \). The sum is to be taken over all relevant modes at frequency \( \omega \). In order to give a full representation of the displacement this sum will normally involve both forward and backward travelling waves (i.e. both positive and negative \( k_r \)). As discussed by Kennett (1984), we can arrange the deepest part of the model of the reference medium so that (5) is a sum over an infinite set of orthogonal modes, and achieve a complete representation of the seismic wavefield. For example, the constructive interference of sufficient numbers of surface-wave modes will synthesize body wave phases (see e.g. the examples in chapter 11 of Kennett 1983). However, especially at higher frequencies, this requires a very large number of modes. In the varying medium we wish to concentrate on the surface-wave field and so would like to be able to work in terms of a restricted (and not too large) set of modes. Such a truncation of the expansion (5) imposes limitations on the size and character of the heterogeneity which can be tackled.

We recall that \( \tau_{11}, \tau_{12} \) are, in general, not continuous across horizontal interfaces and so if the material discontinuities in the heterogeneous model do not coincide with those in the reference model, we have the problem of representing a jump in \( \mathbf{t}_4 \) across an interface by a superposition of continuous traction vectors \( \mathbf{t}^0_{1r} \). A satisfactory fit can be achieved
with many modes, but the accuracy of the representation may be limited with a restricted
mode set (cf. Fourier series). As a result it is desirable that the major discontinuities in both
the heterogeneous and reference models should be coincident.

It should be noted that within the heterogeneity (5) cannot be directly interpreted as a
separation into fundamental and overtone components. The true modal solutions at \( x \), if
they exist, in the heterogeneous medium are likely to have a different character from those
of the reference medium. Only when the heterogeneity is nil, and we have returned to the
reference medium, do the modal coefficients in (5) take on the physical significance of the
actual modal amplitudes. We will consider later the relation of local modal eigenfunctions
to those of the reference structure.

In the reference structure the modal contributions (5) satisfy the coupled equations

\[
\sum_r i k_r \hat{c}_r \left( \frac{w_r^0}{t_{lr}^0} \right) - A^0 \sum_r \hat{c}_r \left( \frac{w_r^0}{t_{lr}^0} \right) = 0
\]

in the absence of any applied force, with coefficients \( \hat{c}_r = c_r \exp(ik_r x) \). We have written
\( A^0 \) to indicate the form of the differential operators for the reference medium.

In a laterally varying medium we have to require the modal coefficients to vary with
position. On expanding the differential operators \( A \) as \( A^0 + \Delta A \), to separate off the
reference medium contribution, we find

\[
\sum_r \left\{ i k_r \hat{c}_r \left( \frac{w_r^0}{t_{lr}^0} \right) + \frac{\partial \hat{c}_r}{\partial x} \left( \frac{w_r^0}{t_{lr}^0} \right) \right\} = (A^0 + \Delta A) \sum_r \hat{c}_r \left( \frac{w_r^0}{t_{lr}^0} \right)
\]

These equations may be rewritten as

\[
\sum_r (i k_r - A^0) \hat{c}_r \left( \frac{w_r^0}{t_{lr}^0} \right) = - \sum_r \frac{\partial \hat{c}_r}{\partial x} \left( \frac{w_r^0}{t_{lr}^0} \right) + \Delta A \sum_r \hat{c}_r \left( \frac{w_r^0}{t_{lr}^0} \right)
\]

in which we can recognize the l.h.s. as having the same form as (6), but we have in addition,
on the right side, a term which is equivalent to a generalized volume force applied to the
reference structure.

As well as the equation of motion we require the wavefield to satisfy certain continuity
conditions at internal and external boundaries where the elastic moduli are discontinuous.
The most general case of an interface in the heterogeneous medium will be titled and
placed from those of the reference medium. Continuity of displacement is assured at
any interface, for a sum of the form

\[
\sum_r \hat{c}_r w_r^0
\]

from the continuity properties of the displacement eigenfunctions. We also require that
the tractions normal to any interface should be continuous. We will assume that all such
surfaces do not vary rapidly in the horizontal direction. Then for an interface in the laterally
varying medium described by the function \( h(x) \), we may work to first order in the slope
\( (h) \), and require using (4)

\[
\left[ (hI + C_{31} C_{11}^-) \sum_r \hat{c}_r(x) t_{lr}^0 + Q \sum_r \hat{c}_r(x) \frac{\partial \hat{c}_r}{\partial x} w_r^0 \right] = 0,
\]

where \( I \) denotes the identity matrix and the square brackets indicate the jump in the
enclosed quantities across the interface, evaluated from bottom to top.
If we now separate out the contribution from the properties of the reference medium from the local variations, we may write

\[
\left[ (C_{31}C_{11}^{-1})^0 \sum_r \hat{c}_r(x) t_{1r}^0 + Q^0 \sum_r \hat{c}_r(x) \partial_z w_r^0 \right]
\]

\[
= - \left[ [\Delta(C_{31}C_{11}^{-1}) + h\mathbb{I}] \sum_r \hat{c}_r(x) t_{1r}^0 + \Delta Q \sum_r \hat{c}_r(x) \partial_z w_r^0 \right].
\]

(10)

The difference terms such as \(\Delta(C_{31}C_{11}^{-1})\) in (10) indicate the discrepancy between the combinations of elastic moduli for the heterogeneous and reference models. In the absence of any lateral heterogeneity the expression on the l.h.s. of (10) would vanish. The presence of the heterogeneity is thus equivalent to inducing a traction discontinuity into the reference medium, along the line of the interface. Such a traction discontinuity is equivalent to a localized volume for along the interface

\[ f^T(x) = - [T] \delta [z - h(x)] \]

(see e.g. section 3.1 of Aki & Richards 1980).

We can therefore represent the combination of the equations of motion and the boundary conditions in the laterally varying medium by means of the following system of equations

\[
\sum_r \hat{c}_r(i k_r - A^0) \left( \begin{array}{c} w_r^0 \\ t_{1r}^0 \end{array} \right) = \sum_r \left\{ - \frac{\partial \hat{c}_r}{\partial x} \left( \begin{array}{c} w_r^0 \\ t_{1r}^0 \end{array} \right) + \hat{c}_r \Delta A \left( \begin{array}{c} w_r^0 \\ t_{1r}^0 \end{array} \right) - \hat{c}_r \left( f_r \right) \right\},
\]

(11)

where the force term is a summation of interface contributions

\[ f_r = \sum_n \left[ \Delta(C_{31}C_{11}^{-1}) + h_n \mathbb{I} \right] t_{1r}^0 + \Delta Q \partial_z w_r^0 \| n \delta [z - h_n(x)]. \]

With the abstraction of this force contribution the interface conditions on the wavefield are reduced to

\[ \left[ \sum_r \hat{c}_r(x) w_r^0 \right]_n = 0, \quad \left[ \sum_r \hat{c}_r(x) \left( (C_{31}C_{11}^{-1})^0 t_{1r}^0 + Q^0 \partial_z w_r^0 \right) \right]_n = 0, \]

where the subscript \(n\) indicates the jump at any interface. These constraints will be automatically satisfied by the representation (5). Equation (11) can be further simplified because the modal eigenfunctions are just the free solutions for the reference structure and so the l.h.s. of the equation vanishes. This leaves us with

\[
\sum_r \frac{\partial \hat{c}_r}{\partial x} \left( \begin{array}{c} w_r^0 \\ t_{1r}^0 \end{array} \right) = \sum_r \hat{c}_r \left\{ \Delta A \left( \begin{array}{c} w_r^0 \\ t_{1r}^0 \end{array} \right) - \left( f_r \right) \right\}.
\]

(12)

We now exploit the orthogonality relation between different modal eigenfunctions for the reference medium

\[ i \int_0^\infty dx \left\{ w_r^0(-k_i, z) \cdot t_{1i}^0(k_j, z) - t_{1i}^0(-k_i, z) \cdot w_r^0(k_j, z) \right\} = \delta_{ij}, \]

(13)

to get a set of coupled first-order differential equations for the modal coefficients \(\{\hat{c}_r\}\), appropriate for both elastic and anelastic heterogeneity. These equations can be written
in the form
\[
\frac{\partial \hat{c}_q}{\partial x} = \sum_r \hat{c}_r \int_0^\infty dz \left\{ \bar{t}_{1q}^0 \cdot \left[ \Delta C_{11} \hat{t}_{1r}^0 + \Delta (C_{111}) \hat{t}_{1r}^0 \right] - \bar{w}_r^0 \cdot \left[ -\Delta \rho \omega^2 \bar{w}_r^0 - \partial_z \left\{ \Delta Q \partial_z \bar{w}_r^0 + \Delta (C_{111}) \hat{t}_{1r}^0 \right\} \right] \right\} + \sum_n \hat{h}_n (\bar{w}_q^0 - \hat{w}_q^0) \hat{t}_{1r}^0_n.
\]

Equation (14) can be simplified by integration by parts, to yield
\[
\frac{\partial \hat{c}_q}{\partial x} = \sum_r \hat{c}_r \int_0^\infty dz \left\{ \bar{t}_{1q}^0 \cdot \Delta C_{11} \hat{t}_{1r}^0 + \Delta \rho \omega^2 \bar{w}_r^0 \cdot w_r^0 \right. \left. - \bar{t}_{1q}^0 \cdot \Delta (C_{111}) \hat{t}_{1r}^0 \right\} \partial_z \bar{w}_q^0 \partial_z \hat{w}_r^0 + \Delta (C_{111}) \hat{t}_{1r}^0 \right\} + \sum_r \hat{c}_r \sum_n \hat{h}_n (\bar{w}_q^0 - \hat{w}_q^0) \hat{t}_{1r}^0_n.
\]

In which we see that there is no explicit interface term in the case of a flat boundary ($\hat{h}_n = 0$). However, the modification of an interface will appear through the integral term in (15), as may be seen from Fig. 1. The shaded zones are ones in which the properties of the laterally varying medium will differ from the reference and for which there will be a significant contribution to the integral which will vary with horizontal position. In a first-order perturbation theory the whole of this effect would be projected onto a specific term, but here we are able to make a more detailed allowance for the behaviour.

The combination of the modal representation (5) with the imposition of continuity of traction at each interface thus leads to the set of coupled equations
\[
\frac{\partial}{\partial x} c_q = i \sum_r K_{qr} \exp (-ik_q x) \exp (ik_r x) c_r,
\]

for the modal coefficients, where the coupling coefficients between the modes can be

Figure 1. A varied interface, the zones of heterogeneity introduced by shifting are indicated by shading. These zones will contribute to the integral in the modal coupling terms.
written as

\[ K_{qr} = \int_0^\infty dz \left[ \mathbf{t}_{1q}^0 \cdot \Delta C_{11} \mathbf{t}_{1r}^0 + \Delta \rho \mathbf{w}_q^0 \cdot \mathbf{w}_r^0 - \partial_z \mathbf{w}_q^0 \cdot \Delta Q \partial_z \mathbf{w}_r^0 \right. \]

\[ \left. - \partial_z \mathbf{w}_q^0 \cdot \Delta (C_{31} C_{11}) \mathbf{t}_{1r}^0 - \mathbf{t}_{1q}^0 \cdot \Delta (C_{11} C_{13}) \partial_z \mathbf{w}_r^0 \right] + \sum_n \hat{h}_n (\mathbf{w}_q^0 \cdot \mathbf{t}_{1r}^0)_n. \]  

Kennett (1984), Kennett & Mykkeltveit (1984) used the integral term in (17) without allowance for the effects of interface slope. However, the numerical implementation of their models retained only horizontal surfaces of discontinuity in the material properties.

The coupled first-order equations (16) are not very easy to solve because we have a two-point boundary value problem with both reflected and transmitted modes to be determined. Kennett (1984, section 3) has shown that an effective procedure is to work with the reflection and transmission matrices for a sequence of models encompassing increasing portions of the heterogeneous medium. This leads to an initial value problem for two coupled-matrix Ricatti equations. These non-linear differential equations for the reflection and transmission matrices can be readily solved numerically.

3 Restrictions on heterogeneous models

The essence of this coupled mode technique is that we can represent the displacement and traction fields as a superposition of the modal eigenfunctions for the reference model. For practical applications of the coupled-mode scheme we would wish to use a limited number of modes, but we cannot rely on the same criteria for mode selection as would be sufficient for dispersion analysis. The modal representation can only be effective if we use a sufficiently large number of modes with diverse behaviour so that the modal set is able to represent as large a class of functions as possible. The problem is the same as in Fourier series representation of functions: we can only expect to give a good account of the nature of a function if sufficient Fourier components are included. The most rapid allowable variation in the displacement and traction fields is dictated by the last mode retained in the truncated modal set.

This style of coupled-mode treatment can only be expected to be effective when the heterogeneous model does not deviate too strongly from the reference model. We will now try to specify the restrictions which we need to place on the model, and the number of modes to be included to achieve a satisfactory representation of the wavefield.

GEOMETRIC RESTRICTIONS

When we work with a fixed reference structure we can only consider spatially limited regions of heterogeneity. In order to give a direct physical interpretation of the modal coefficients, we must require the surface wave-train to start and finish in the reference model. This restriction is somewhat limiting as to the range of possible applications, e.g. we cannot examine a major horizontal change in structure. However, there is a major computational compensation since only one set of modal eigenfunctions need be computed. It is possible to extend the treatment to a variable background (Maupin 1987) but then, at the very least, modal eigenfunctions have to be determined for a number of different stratified models. Such a scheme differs from the 'local mode' scheme employed by Odom (1986) because the varying reference structure may differ from the local structure.
In models containing major interfaces, we have noted above that the basic modal representation is likely to be of relatively limited validity if these interfaces are displaced between the heterogeneous and reference models, unless the number of modes used is large. For an interface between two solids, we have discontinuities in the $\tau_{11}$, $\tau_{12}$ components but these are not normally too severe unless there is a very major change of properties, e.g. sediments against hard rock.

The problems are compounded for a fluid/solid boundary. In the fluid $\tau_{13}$, $\tau_{23}$ are zero, so there is a large jump in traction on entering the solid. The horizontal displacements $w_1$, $w_2$ are discontinuous across a horizontal interface. If a fluid/solid boundary is displaced, the conditions at the vertical interface where we weld on the reference structure no longer show continuity of traction $t_1$ and displacement over the whole depth range: for, where solid and fluid media abut on a vertical interface, $\tau_{13}$ should be zero and $w_3$ may be discontinuous. Therefore, a modal sum using the reference eigenfunctions is inadequate however many modes are employed. In consequence, using a fixed reference structure and a limited number of modes, we can only obtain accurate results for oceanic models when there are no variations in water depth or sediment thickness.

![Figure 2](image_url)

**Figure 2.** Schematic illustration of the coupling between different modes introduced by heterogeneity. The locations of the first 10 Rayleigh modes for a continental model at 20 s period are shown on the wavenumber axis. The wavenumber spectrum of various Gaussian perturbations with the same maximum deviations from the reference model are superimposed for different scale parameters $\sigma$. --- 125 km, -- - 250 km, — — — 500 km, ——— 1000 km. In order to allow an assessment of scale-length the horizontal ordinate $\hat{k} (=k/2\pi)$ has dimensions km$^{-1}$.

**Horizontal Scale Lengths**

We can get an idea of how the nature of the heterogeneity interacts with the number of modes included in the calculation by looking at the horizontal and vertical spatial wave-numbers in both the modal eigenfunctions and the heterogeneity.

Kennett (1972) introduced a representation for seismic wave scattering in the horizontal wavenumber domain and showed that, to first order, the effect of bounded heterogeneity was convolutional. Consider a deviation from the reference model with a wavenumber spectrum $f(k, z)$. Then for an incident mode with wavenumber $k_i$, the first-order scattered
wave contribution to the surface wave field would be of the form

$$\sum_r \int dz \ f(k_r - k_f, z) \ P_r(k_r, z),$$

(18)

where $P_r$ represents a complex set of propagation terms. Higher-order scattering will broaden the range of coupling but (18) allows a convenient semi-quantitative measure of horizontal wavenumber coupling.

If we assume that we have the same amplitude spectrum at each depth, then we can visualize the way in which the energy is spread across the different modes by superimposing the shape of the shifted spectrum $f(k - k_f)$ on the location of the modal poles as is done in Fig. 2. For a period of 20 s, we consider scattering from the second higher Rayleigh mode into all modes with phase velocity greater than 7 km s$^{-1}$ for the continental model illustrated in Fig. 3. We show the wavenumber spectra for a consequence of Gaussian shaped deviations from the reference model $\exp(-x^2/2\sigma^2)$ with a sequence of different scale-lengths. The same maximum heterogeneity is assumed in each case, so that the wavenumber spectrum becomes more sharply peaked for larger scale heterogeneity. For scales of the order of 2500 km, the decay of $f$ away from $k_2$ is so sharp that there is no intermode coupling, and the non-interacting local-mode treatment of Woodhouse (1974) will be effective. For shorter length-scales, the second higher mode will interact with a number of other modes; indeed for $\sigma = 125$ km, the coupling range extends outside the group of modes considered. Rapidly varying heterogeneity of any appreciable magnitude is likely therefore to make it difficult to work with just a limited set of modes. The scattering estimates provided by (15) are global in character, rather than being local as in the treatment of section 2, but give us a good general guide as to likely behaviour.

**Vertical Scale Lengths**

We now turn our attention to the vertical variations in the heterogeneity. The presence of deviations from the reference model leads to coupling between modes determined by the integrals $K_{qr}$ (17) over the modal eigenfunctions and the heterogeneity terms. In terms of a vertical wavenumber decomposition the behaviour is similar to the horizontal case, coupling between modes will occur when the wavenumber difference between modes matches the wavenumber content of the heterogeneity. As it is not easy to visualize the vertical wavenumber content of an eigenfunction, we will illustrate the depth dependence by reference to specific examples.

For a given frequency the number of zero crossings in the modal eigenfunctions increases with mode number and, in general, the modes penetrate to greater depth. As a result of the balance between these two factors, there is a steady progression of the vertical wavenumber content of the eigenfunctions with mode number. Such a situation occurs, for example, for Rayleigh-wave overtones propagating in both oceanic and continental mantle models (Fig. 3). For an increasing velocity with depth, the behaviour near the deepest maximum of each eigencomponent is very similar for each mode but moves to greater depth as the mode number increases. On the other hand, where there is a well-defined waveguide with sharp boundaries such as the crust for high frequency $L_g$, there will be a steady progression in modal character until the mode number is such that the waves are no longer confined to the waveguide. At that point a new sequence with rather larger vertical scales will be set up in the structure beneath the waveguide (see e.g. fig. 3 of Kennett 1984). It is rather hard to assess the vertical wavenumber content of the fundamental and first higher modes of both Love and Rayleigh waves because of their rapid decay with depth and weakly...
Rayleigh Modes  Period: 20 s

Figure 3. The modal eigenfunctions for the first nine overtones of the Rayleigh modes at 20 s period for the continental velocity model shown at the right after earth flattening. The phase-velocity limit for this mode set was 7 km s⁻¹. The modal set provides very good coverage of the region down to 400 km.

oscillatory behaviour. In this case we can see from the behaviour of the eigenfunctions that interactions with other modes will be most significant for heterogeneity in the near-surface zone. Since all the modes tend to have noticeable displacements, similar in shape, in this region, it is possible for coupling to occur from the fundamental and first higher modes into a relatively broad span of higher modes.

With any truncated modal set there will be some types of wave propagation with large phase velocities (and thus steep angles of propagation) which cannot be represented. We therefore would want to avoid heterogeneity with 'sharp corners', which can generate diffractions or other steeply propagating waves. In this context, we should note that perturbation of discontinuities is equivalent to introducing a sharply bounded heterogeneity and is likely to give rise to problems unless an extensive modal set is used. In general, we would be wise to avoid any form of heterogeneity with changes which are rapid compared with the scale of the region of interest. For example, for the mantle down to 400 km at 20 s period (as illustrated in Fig. 3), it would be difficult to ensure a good representation of the effects of significant heterogeneity with a vertical scale much less than 50 km.

Behaviour of Modal Expansions

It is difficult to draw general conclusions on the behaviour of truncated modal expansions because of the dependence on the particular character of the heterogeneity field. However, we can get some insight into the likely degree of coupling between modes by looking
Modal expansion for varied model

Figure 4. Illustration of the expansion of the modal eigenfunctions for a varied model in terms of those for a reference structure. The expansion coefficient is shown as a function of difference in mode number, with the range of behaviour over 8 modes indicated by the shaded regions. The spherical velocity distributions for the reference model and the varied model, indicated by the chain dotted line, are shown in the insert.

at the representation of the eigenfunctions of a varied structure in terms of those of the reference model.

As an illustration we show in Fig. 4 the expansion of the modal eigenfunctions for an upper mantle model with a pronounced low-velocity zone in terms of the reference eigenfunctions illustrated in Fig. 3 at 20 s period. The two upper-mantle models have a similar trend in velocity gradient below 200 km but differ significantly in velocity. The expansion coefficients are shown as a function of the difference in mode number for the first eight higher modes, with the range of variation shown by shading. The dominant expansion coefficients occur over a band of four modes with the main contribution concentrated on the corresponding mode in the reference structure. Thus in order to describe propagation of a single mode in the varied structure at least four modes would be coupled together in the modal representation (5).

The pattern of behaviour seen in Fig. 4 is typical of a situation where the main difference in the models is in the value of the seismic velocities. As the mode number increases there is a tendency for the coefficient with no mode-number difference to diminish whilst the magnitude of the coefficients with a single mode-shift increases.

In the case of a shift in the level of a major discontinuity in the seismic parameters, the behaviour of the modal expansion coefficients is rather different. Away from the central mode, the sign of the coefficients alternates with mode-number difference and the rate of decay of the modal expansion is much slower. This arises because the structure of the dispersion equations and, therefore the nature of the modal eigenfunctions, is much
Truncated modal expansions in laterally varying media

Figure 5. Sketch of the structure of the coupling matrix $K$ in the presence of two weakly coupled waveguides. The dominant terms are usually on the diagonal. There is often a zone of overlap between the effects of two waveguides, where modes have transitional characters and so can couple to either group depending on the nature of the heterogeneity. The chain dotted line shows how one might truncate the modal set to preserve the mode coupling in the first group.

more sensitive to the position of major discontinuity than to the details of the velocity distribution between them.

CHOICE OF NUMBER OF MODES

For many problems, the nature of the velocity model is such that the propagation is associated with a number of loosely coupled waveguides, as for example in the case of the crust and the upper mantle. The character of the eigenfunctions also tend to mirror such divisions, and so when heterogeneity is introduced coupling between the waveguides is restricted. The typical form of the coupling matrix $K$ in such a case is sketched in Fig. 5, we get two dominant blocks of coupling overlapping in the region of modes of transitional character. This structure arises because the effect of heterogeneity in a waveguide is to induce preferential coupling between modes with their dominant energy in that waveguide.

When we seek to truncate the modal set we should aim to preserve this block character, and also to allow for some interaction between blocks by cutting with a margin of 2–3 modes, as indicated by the chain dotted line in Fig. 5. The coupling between the modes of major interest is then retained at the expense of a poor representation of deeper propagation. Numerical experiments with various levels of modal truncation indicate that, unless the heterogeneity is small, the effect of removing a further mode has a significant effect on calculated reflection and transmission matrices. The errors concentrate on the coupling terms for the highest remaining mode but extend out to neighbouring modes. As a result, at least two more modes should be included than would be suggested by the phase-velocity content of the wavetrain to be studied.

One of the best types of heterogeneous model for use with a limited mode set is one in which the deviations from the reference model are of a roughly random distribution of positive and negative excursions and occur on a medium scale (e.g. around 100–200 km for the case shown in fig. 3). In this case quite large deviations can be tolerated (at least
±3 per cent); scattering occurs into a broad band of modes, but the absolute level of coupling is not too large.

For an upper-mantle problem of this type, with periods greater than around 20 s, velocity variations less than 5 per cent and perturbations in major interfaces of less than 10 km, we can hope to get a good representation of the behaviour of the first 5 modes if a total of, at least, 8 modes are retained. For the lower modes the transmission terms in the same mode will be accurate to about 1 per cent over a horizontal propagation distance of around 2000 km. The errors in cross-coupling terms will be of the same size but relatively more important because of the smaller size of these terms. However, the cross-coupling contribution to the seismogram will not be very large and so the overall error will be at most a few per cent. This compares favourably with typical errors in the analysis of observed seismograms in terms of higher modes.

Reflected waves

Reflected surface waves are generally produced from relatively rapid horizontal changes in seismic properties on the scale of the wavelengths of surface wave-modes. Such features are also likely to generate body waves by diffraction or scattering and these may be lost to a calculation in terms of coupled modes unless a sufficiently large suite of modes is considered.

For heterogeneity in the upper parts of the model it is often possible to give a good account of the reflected waves. For example, for crustal heterogeneity a mode set which includes phase velocities up to 5–6 km s\(^{-1}\) will normally give good results. At high frequencies (0.5 Hz and above) this requires that modes in both the Lg and Sn blocks are included. For the deeper regions of the models significant reflected waves are, in general, associated with departures from the reference model at the limit of what can reasonably be accommodated by the present method.

In the absence of features with strong horizontal gradients in properties, most of the mode coupling occurs in transmission. It is therefore tempting to consider the neglect of backward travelling waves in our local modal sums as we carry the representation (5) through the heterogeneous zone. Such an approximation unfortunately often gives rise to noticeable errors. In regions with systematic deviations from the reference model, apparently reflected waves may need to be introduced to improve the match to the wavefield behaviour, even though for the total model the overall reflected wave is small. There is a close analogy with the need to use both positive and negative wavenumbers in a conventional Fourier transform.

Acknowledgment

VM thanks the Research School of Earth Sciences at the Australian National University for hospitality during much of the period during which this work was undertaken.

References


Stacking gravity tide observations in central Europe for the retrieval of the complex eigenfrequency of the nearly diurnal free-wobble

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Accepted 1987 April 28. Received 1987 April 6; in original form 1986 August 20

Summary. We have used tidal gravity measurements from six stations in central Europe to investigate the resonance in the diurnal tidal band, caused by inertial coupling between the mantle and outer core of the Earth. By the use of stacking it was possible to determine the eigenfrequency and quality factor of this eigenmode, commonly called the 'nearly diurnal free-wobble'. We assessed the effect of systematic errors from the ocean correction to the tidal measurements employing a Monte-Carlo method. The observed eigenfrequency is $1 + 1/(434 \pm 7)$ cycles per sidereal day, and is significantly higher than predicted by theories. The observed quality factor is $(2.8 \pm 0.5) \times 10^3$.

Key words: wobble, diurnal resonance, gravity tides, complex eigenfrequency.

1 Introduction

It is well known that the existence of the Earth’s fluid core within an elastic mantle confined by the elliptical core–mantle boundary leads to a second eigenmode in the rotational spectrum besides the Chandler wobble (e.g. Toomre 1974). On the one hand this mode can be described as a retrograde rotation of the instantaneous rotation axis of the Earth relative to the figure axis, this is the nearly diurnal free wobble (NDFW) with an eigenperiod close to one sidereal day in a frame rotating with the Earth. On the other hand this mode involves a motion of the instantaneous rotation axis relative to the direction of angular momentum; this is the associated free core nutation (FCN) with an amplitude about 460 times larger than the wobble and a theoretical period of about 460 sidereal days as seen from inertial space.

There are two different situations in which eigenfrequencies of a physical system can be
determined. On the one hand, the system can be ‘excited’ by sources having a spectrum including the eigenfrequency of the system. The system oscillates at its eigenfrequency which can then be measured directly. Corresponding examples are the seismic normal modes excited by earthquakes and the Chandler wobble, the excitation sources of which are still being discussed (Chao 1983, 1985). Because of the simplicity of the source time function, the excitation of the seismic normal modes does not, at least to first order, interfere with the measurement of $Q$. The case for the Chandler wobble is much more complicated. On the other hand, a system can be ‘forced’ harmonically at frequencies near the eigenfrequency in which case a resonant response of the system to the driving forces is observed and the parameters of this mode can be inferred from the resonant behaviour of the system. This is the case we are dealing with here: the tidal forcing of the Earth in the vicinity of the FCN/NDFW eigenmode leads to an indirect effect on the observed amplitudes and phases of the forced nutations and Earth tides, respectively. In the following we refer to this eigenmode as the NDFW, because we are mainly dealing with this aspect of the mode in this paper.

The NDFW as an eigenmode of the Earth was predicted long ago (e.g. Hough 1895; Poincaré 1910). The direct astronomical observation of the FCN has been a controversial matter for a long time (e.g. Sasao & Wahr 1981). More recently precise nutation measurements using VLBI techniques set a firm upper bound on the amplitude (Eubanks, Steppe & Sovers 1986; Herring, Gwinn & Shapiro 1986: Robertson, Carter & Wahr 1986). This is a very recent development while attempts to identify the resonance in Earth tide measurements started about 20 years ago (Melchior 1966). These first attempts were characterized by comparison of observed amplitude ratios of large tidal constituents $O_1$, $P_1$, $K_1$ with theoretical ratios. Little was known or done then to correct the observed amplitudes for the effects of oceans and atmosphere. With higher quality data becoming available, the next generation of observations dealt additionally with the measurement of minor tidal constituents $\psi_1$, $\phi_1$ much closer to the predicted resonance than $K_1$. Among the best observations of this kind are those by Abours & Lecolazet (1979) and Levine (1978) using gravity and strain data, respectively. Lecolazet (1983) correlated observed amplitude variations with variations of the length-of-day and speculated about a temporal shift of the NDFW eigenfrequency.

With the advent of the superconducting gravimeter and an improved understanding of atmospheric and oceanic effects, the first attempts were made to determine the eigen-frequency and $Q$ of the mode (Warburton & Goodkind 1978; Goodkind 1983) and to provide new information on the theory from observations. A major problem for the retrieval of the parameters of this mode is the accuracy of oceanic corrections to the data; these corrections are rather large in California (7 per cent of body tides), where the first superconducting gravimeters were located. Several more recent investigations dealt with different data sets from Central Europe (Hinderer et al. 1986; Neuberg & Zürn 1986; Zürn, Rydelek & Richter 1986). Neuberg & Zürn (1986) suggested a stacking method for data from different instruments at a single station.

From all these investigations it became clear that oceanic contributions and systematic errors arising from the corresponding corrections play the most important role in the error budget. We therefore conjectured that gravity-tide observations in central Europe should provide superior estimates for the NDFW parameters for the following reasons:

1. Ocean load effects on gravity-tides in central Europe are known to be very small for diurnal tides (0.5 per cent of body tides). It is very likely that systematic errors in the corrections should be rather small as well (e.g. Souriau 1979; Gerstenecker & Varga 1986).

2. A number of long high quality data sets are available as a basis for the analysis including the two superconducting gravimeters at Frankfurt and Bruxelles.
Complex eigenfrequency of the NDFW

(3) The atmospheric effect on gravity tides can be modelled much better than on any other tidal component (Warburton & Goodkind 1978; Rabbel & Zschau 1985).

(4) Using a stacking method similar to Neuberg & Zurn (1986), all data sets can be analysed simultaneously in order to reduce the random errors.

In the next section the theoretical background will be presented, in Section 3 the data and the reduction of the data will be described. The model, a harmonic oscillator, will be introduced in Section 4. Section 5 describes the actual methods of analysis. An attempt to assess realistically the systematic errors from the ocean loading corrections will be described in Section 6. Finally the last two sections will present the results and some speculations about physical effects on the frequency, $Q$, and the strength of the mode.

2 Theoretical background

In this section, we develop a theoretical expression for the tidal change in gravity $\Delta g$ of an earth model composed of a fluid core and a deformable mantle submitted to the diurnal part of the luni-solar tidal potential. We first consider the elasto-gravitational deformation of the Earth due to this potential. We use then the Euler equations for conservation of angular momentum; on the one hand, without tidal forcing, in order to obtain the complex eigenfrequencies of the rotational modes; on the other hand in order to express the forced resonant rotational motions of the core and the whole Earth as a function of the luni-solar potential. The misalignment of the rotation axes of core and mantle will give rise to a tesseral pressure distribution at the elliptical core–mantle boundary (CMB). When this is calculated, we specify then the resonant elastic deformation and the resulting resonant tidal gravity change.

2.1 Elasto–Gravitational Deformation

The elastic deformation of a gravitating earth is considered here in the static and linearized case (to first order with respect to the displacement $u$, density perturbation $\rho'$ and gravitational potential of mass redistribution $V'$) assuming spherical symmetry and hydrostatic prestress.

The perturbations due to the ellipticity and rotation (e.g. Coriolis force) are found to be of the order of 1 per cent (Wahr 1981).

Navier's elastostatic equations are (Alterman et al. 1959):

$$\nabla \cdot \overline{\tau} + \nabla (\rho_0 u \cdot \nabla V_0) + \rho' \nabla V_0 + \rho_0 \nabla V' + \rho_0 \nabla V = 0. \quad (1)$$

where $\overline{\tau}$ is the stress tensor related to the displacement $u$ by Hooke's law in the perfect linear elastic case.

The gravitational potential $V_0$ before deformation is related to the density $\rho_0$ before deformation by Poisson's law, and similarly for the perturbations after deformation $V'$ and $\rho'$; besides, $\rho'$ must satisfy the continuity equation.

In the static and spherical approximation, there is no coupling between the toroidal and spheroidal modes of deformation and the integration of the equations (1) can be achieved separately. The boundary conditions at the CMB $r = b$ are the usual ones (see e.g. Dahlen 1974) modified, in our case, by the hydrodynamical pressure $P_n$ (coefficient of the spherical harmonic development in $r/b^n$, where $n$ is the order) arising from the differential core rotation. Elastic effects due to tangential tractions (of viscomagnetic origin for instance) are neglected.

The solutions to this problem are given in the form of generalized 'Love numbers' (e.g.
Legros, Amalvict & Hinderer 1986; Legros & Amalvict 1987):

\[ u_r(a) = h_n \frac{V_n}{g} + \tilde{h}_n \frac{P_n}{\rho g} \]

\[ u_t(a) = l_n \frac{V_n}{g} + \tilde{l}_n \frac{P_n}{\rho g} \]

\[ V'(a) = k_n V_n + \tilde{k}_n \frac{P_n}{\rho} \]

where \( V_n \) is the coefficient of the spherical harmonic development into \((r/a)^n\) of the volume potential (e.g. tidal, centrifugal), which is acting on the whole Earth, \( a \) is the radius of the Earth, \( u_r(a), u_t(a) \), the radial and tangential displacement and \( V'(a) \) the potential of mass redistribution. \( h_n, l_n, k_n \) are the classical Love numbers and \( \tilde{h}_n, \tilde{l}_n, \tilde{k}_n \), internal pressure (at CMB) 'Love numbers' of order \( n \) (independent of the degree \( m \) for a given order \( n \) for symmetry reasons). With our definition of the 'Love numbers', \( \rho \) is the mean density of the mantle and \( g = g(a) \) the mean surface gravity.

In order to calculate the changes of the core inertia tensor for a homogeneous core and spherical symmetry (with respect to elastic deformation), we only need the radial displacement \( u_r \) (Sasao, Okamoto & Sakai 1977) of the CMB, which becomes now:

\[ u_r(b) = h_n(b) \frac{V_n}{g} + \tilde{h}_n(b) \frac{P_n}{\rho g} \]

with the help of the quantities (akin to Love numbers) \( h_n(b) \) and \( \tilde{h}_n(b) \). In the following we restrict the analysis to order \( n = 2 \) and omit this subscript.

A more complete expression for the elastic yielding taking into account the existence of a superficial fluid layer (oceanic or atmospheric) and additional phenomena can be found elsewhere (Legros et al. 1986; Legros & Amalvict 1985).

2.2 NEARLY DIURNAL RESONANCE

2.2.1 Euler equations

The rotational motion of a two-layer earth model composed of a liquid core and elastic mantle can be described by the Euler equations for conservation of angular momentum.

Correct to the first order, the developed form of these equations for the equatorial components of rotation becomes in the Tisserand frame of the mantle mean axes (for more details, see e.g. Sasao, Okubo & Saito 1980, Hinderer, Legros & Amalvict 1982, 1987).

\[
\dot{\omega} (1 + ak/k_s) - i\Omega \omega (1 - k/k_s) + (\dot{\omega}^c + i\Omega \omega^c) (A^c/A + ak/k_s)
\]

\[
= (3ak) (k_s a^2)^{1/2} (W/\Omega + iW) - 3i\alpha W/a^2 \quad \text{(4a)}
\]

\[
\dot{\omega} (1 + q_0 h^c/2) + \dot{\omega}^c (1 - q_0 h^c_2/2) + i\Omega \omega^c (1 + \dot{\alpha}^c + K' - iK)
\]

\[
= (3q_0 h^c (2a^2)^{1/2} (W/\Omega). \quad \text{(4b)}
\]

These are the Liouville equations for the whole Earth (4a) and for the core (4b).

We use a complex notation for the Earth's and core wobble components \( \omega = \omega_1 + i\omega_2 \), \( \omega^c = \omega_1^c + i\omega_2^c \) and tidal potential \( W = W_1 + i\tilde{W}_1 \), where \( W_1, \tilde{W}_1 \) are the components of the tesseral part of order 2 and degree 1 (see equation 2.19 of Hinderer et al. 1982), \( k \),
The eigenfrequencies of this system can be found by setting $W = 0$ (absence of tidal potential):

$$\sigma_{\text{nd}} = -\Omega \left[ 1 + (A/A^m)(\alpha^c - q_0 h_f^i/2 + K' - iK) \right]$$

$$\sigma_{\text{cw}} = (A/A^m) \alpha (1 - k/k_s) \Omega.$$  (5)

The subscript nd stands for NDFW, cw for Chandler wobble. The period of the FCN associated with the NDFW is $2\pi/|\Omega + \sigma_{\text{nd}}|$. $A^m = A - A^c$ is the mantle moment of inertia. The term $q_0 h_f^i/2$ arises from the deformation of the CMB due to elasticity.

Because of the small values of the coupling constants $K$ and $K'$, probably of order $10^{-7}$ with the usual estimates of the physical parameters near the CMB, there is no perturbation of the Chandler frequency $\sigma_{\text{cw}}$, to the chosen order of approximation, due to the visco-magnetic torque and hence no damping of this mode.

It is clear that anelastic properties within the mantle (e.g. Zschau 1978; Anderson & Minster 1979; Smith & Dahlen 1981; Okubo 1982) or oceanic loading and friction effects (Dickman 1983) are able to induce a damping mechanism for both eigenmodes which is not taken into account in (5).

2.2.3 Tidally forced rotational motions

With the help of the Euler equations (4), one obtains the amplitude of the Earth's and core rotations $\omega$, $\omega^c$ as a function of the tidal potential $W$. Setting $\omega = \omega_0 \exp (i\alpha t)$, $\omega^c = \omega_0^c \exp (i\alpha t)$ and $W = W_0 \exp (i\alpha t)$, the rotational responses $\omega_0$ and $\omega_0^c$ become for $|(\alpha + \Omega)/\Omega| \ll 1$ (nearly diurnal tidal excitation) to the main order (see e.g. Hinderer 1986):

$$\omega_0 = -\frac{A (1 - q_0 h_c^i/2\alpha A) + \Omega (\alpha^c - q_0 h_f^i/2)}{A^m (\sigma - \sigma_{\text{nd}}) \Omega a^2} 3\alpha W_0$$

$$\omega_0^c = \frac{A (1 - q_0 h_c^i/2\alpha a) 3\alpha W_0}{A^m (\sigma - \sigma_{\text{nd}}) a^2}.$$  (6)

We immediately see the resonance effect in $(\sigma - \sigma_{\text{nd}})^{-1}$ appearing in the rotational motions for a given potential of amplitude $W_0$ and frequency $\alpha$. 
The slight damping introduced by the imaginary part of the eigenfrequency is able to cause a phase lead for some luni-solar nutations as pointed out by Toomre (1966).

Notice that, because of the assumption \(|(\alpha + \Omega)/5\Omega| < 1\), \(\alpha^c, q_0 < 1\) the core response \(\omega^c_0 > \omega_0\) by one order of magnitude.

2.2.4 Tidal gravity change

The predominant core rotation \(\omega^c_0\) will lead to a fluid overpressure \(P = \rho^c b^2 \Omega \omega^c_0 / 3\) acting at the CMB \((r = b)\) and, when substituting \(P\) into (2), the resulting elasto-gravitatianal deformation is then specified by the following formulae (e.g. Hinderer 1986):

\[
\begin{align*}
\phi_T(a) &= \left( h_1 + h_1 \frac{c_{nd}}{\sigma - \sigma_{nd}} \right) (W_2/g) = h^* (W_2/g) \\
\phi_T(a) &= \left( l_1 + l_1 \frac{c_{nd}}{\sigma - \sigma_{nd}} \right) (W_2/g) = l^* (W_2/g) \\
\phi_T(a) &= \left( k + k_1 \frac{c_{nd}}{\sigma - \sigma_{nd}} \right) W_2 = k^* W_2.
\end{align*}
\]

where \(c_{nd} = \Omega (A/A^m) (\alpha - q_0 h^c/2)\); \(h_1, l_1, k_1\) are \(h, l, k, (n = 2)\), respectively, multiplied by the constant quantity \((\rho^c/\rho)(b/a)^2\).

It is clear that the nearly diurnal resonance will be present in every observable quantity representing the response of the Earth to diurnal tidal forcing.

We consider only the resonant tidal gravity change at the surface \(r = a\) resulting from the equations (7):

\[
\Delta g(a) = -\delta^* (2W_2/a)
\]

with

\[
\delta^* = (1 + h - 3k/2 + \frac{A (h_1 - 3k_1/2) (\alpha - q_0 h^c/2) \Omega}{A^m (\sigma - \sigma_{nd})})
\]

The theoretical gravity variation consists of two terms: the usual (static) gravimetric factor \(\delta = 1 + h - 3k/2\) and a dynamical contribution showing the core resonance for some tidal waves in the diurnal band of frequency \(\sigma\) close to the NDFW-eigenfrequency \(\sigma_{nd}\).

The frequency dependence of the Love numbers shown above expresses the resonant behaviour of the deformation, where of course inertial forces and anelasticity in (1) have been completely neglected. Anelasticity in the mantle can formally be included in the results by making the various ‘Love numbers’ complex quantities. An extension of formula (8) taking into account the additional resonant contribution coming from an oceanic layer can be found elsewhere (Wahr & Sasao 1981; Hinderer et al. 1986). The strength of the resonance \(A^* = \Omega A (h_1 - 3k_1/2) (\alpha - q_0 h^c/2)/A^m\) in gravity tides involves parameters which are dependent either on some geometrical and dynamical properties of the model \((\alpha, q_0, A/A^m)\) or its elastic behaviour \((h_1, k_1, h^c)\).

3 Data and data reduction

For this investigation we used tidal admittances of gravity data only, which have been derived from time series recorded at six stations in central Europe: Berlin, Potsdam, Bruxelles, Frankfurt, Schiltach and Strasbourg. Since we intend to apply a stacking method in order to obtain global parameters, it would be best to use as many randomly distributed
tidal stations as possible. In that case, random errors would tend to cancel. We have, on the other hand, the advantage of using just a few stations, but all located in central Europe, where the influence of oceans — the most critical source of error — is small for gravity data.

We gathered from literature the tidal admittances for Berlin (Asch et al. 1986), Potsdam (Dittfeld 1985), Bruxelles (Ducarme, van Ruymbeke & Poitevin 1986) and Frankfurt (Zürn et al. 1986). The results from Strasbourg were given to us by R. Lecolazet, the time series recorded in Schiltach was analysed by us and both results are listed in Table 1. Some information about tidal observations used in this investigation is summarized in Table 2.

Different treatments of barometric pressure effects are involved in the different data sets. The Berlin and Postdam data were not corrected at all, while for Strasbourg and Bruxelles data experimental pressure coefficients were used to correct for this effect. The data from Frankfurt and Schiltach were analysed with the multi-channel HYCON-program (Schüler 1986) which takes barometric effects into account simultaneously with the actual tidal analysis.

The tidal admittance that can be expressed for any tidal frequency by either an amplitude ratio (e.g. gravimetric factor) and phase shift or as a complex quantity can be described essentially as a normalized response of the Earth to the driving tidal forces. Therefore we consider that quantity a sum of several responding contributions, one of which is the resonance effect, appearing at tidal frequencies in the diurnal band. The purpose of data reduction is now to separate the resonance contribution for the tidal constituents $P_1$, $K_1$, $\psi_1$ and $\phi_1$. As a reference admittance, at a frequency at which the resonance effect is supposed to contribute only slightly, we use the tidal admittance for $O_1$.

In a first step, we apply the ocean load-corrections which were computed from the available ocean tidal maps of Schwiderski (1980) for $O_1$, $P_1$ and $K_1$. To obtain the corrections for

---

**Table 1.** Tidal admittances of gravity data recorded in Schiltach and Strasbourg.

**Strasbourg:**

<table>
<thead>
<tr>
<th>Tide</th>
<th>Gravimetric factor</th>
<th>Phase lead [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>1.1474</td>
<td>0.0005</td>
</tr>
<tr>
<td>$P_1$</td>
<td>1.1496</td>
<td>0.0009</td>
</tr>
<tr>
<td>$K_1$</td>
<td>1.1354</td>
<td>0.0007</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>1.2409</td>
<td>0.0410</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1.1571</td>
<td>0.0222</td>
</tr>
</tbody>
</table>

**Schiltach:**

<table>
<thead>
<tr>
<th>Tide</th>
<th>Gravimetric factor</th>
<th>Phase lead [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>1.1489</td>
<td>0.0006</td>
</tr>
<tr>
<td>$P_1$</td>
<td>1.1484</td>
<td>0.0011</td>
</tr>
<tr>
<td>$K_1$</td>
<td>1.1345</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>1.2157</td>
<td>0.0473</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1.1632</td>
<td>0.0260</td>
</tr>
</tbody>
</table>
### Table 2. Information about the time series from which tidal admittances are derived. \( e(P_j) \) is the standard deviation (in \( \% \)) of the gravimetric factor for \( P_j \), estimated from tidal analysis and listed in order to compare the different noise levels of the time series (esf = electrostatic feedback).

<table>
<thead>
<tr>
<th>Tidal Station</th>
<th>Latitude</th>
<th>Longitude</th>
<th>Length of time series</th>
<th>Standard deviation ( e(P_j)/% )</th>
<th>Gravimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Berlin</td>
<td>52.457</td>
<td>13.354</td>
<td>13.3</td>
<td>0.447</td>
<td>LaCoste Romberg ET18</td>
</tr>
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<td>86</td>
<td>0.067</td>
<td>Askania GS15</td>
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<tr>
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<td>50.809</td>
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<td>39.4</td>
<td>0.035</td>
<td>Superconducting</td>
</tr>
<tr>
<td>Frankfurt</td>
<td>50.229</td>
<td>8.611</td>
<td>36</td>
<td>0.017</td>
<td>Superconducting</td>
</tr>
<tr>
<td>Schiltach</td>
<td>48.330</td>
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<td>22</td>
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<tr>
<td>Strasbourg</td>
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<td>7.684</td>
<td>90</td>
<td>0.078</td>
<td>LaCoste Romberg ET8 (esf)</td>
</tr>
</tbody>
</table>

\( \psi_1 \) and \( \phi_1 \), for which no co-tidal maps are available, we followed a suggestion by Wahr (1983) and scaled the \( K_1 \)-correction using the ratio of the tidal potentials \( \psi_1/K_1 \) and \( \phi_1/K_1 \), respectively. By multiplying this quantity by the ratio of the corresponding theoretical diminishing factors \( \gamma(\psi_1)/\gamma(K_1) \) or \( \gamma(\phi_1)/\gamma(K_1) \), we include already the resonance effect in the oceans. As shown by Zürn et al. (1986) the corrections for \( \psi_1 \) and \( \phi_1 \) have very little influence on the results.

As mentioned before, we suppose the ocean correction to be the most essential uncertainty involved in the data reduction. Therefore we investigated the influence of the uncertainty in the ocean load-correction on the resulting quality factor and eigenfrequency by a method described in Section 6.

In a second step, we remove all constant frequency-independent contributions. Assuming that tidal constituent \( O_1 \) is only slightly affected by the resonance, its tidal admittance essentially consists of any frequency-independent contributions, which all other diurnal tidal admittances also have. Hence, subtracting from all other admittances that of \( O_1 \), corrected for ocean loading, removes the constant frequency-independent part from all other admittances. Now we have the differences of observed tidal admittances corrected for ocean loading at frequencies \( \sigma \) to the same quantity at reference frequency \( \sigma_{ref} \). These values are supposed to represent the isolated effect of resonance plus noise. We therefore refer to it in the following as the resonant admittance for brevity and denote it by \( R \). The real part of this complex quantity is in phase with the tidal potential. Goodkind (1983) and Neuberg \& Zürn (1986) used the term ‘load’; however, this could lead to some confusion. We note here that uncertainties in the absolute calibration of the different instruments (of the order of less than 1 per cent) are largely eliminated as a source of systematic errors because of the adopted procedure.

### 4 Model

To determine the resonance parameters we compare the resonant admittances \( R \) with an appropriate model, where these parameters are explicitly involved.
Equations (4) present a set of four coupled differential equations of first order for the variables \( \omega_1, \omega_2, \omega_1^* \) and \( \omega_2^* \). Those equations can be transformed into four decoupled differential equations of fourth order, for each of these variables. The four differential operators applied to the four variables in these equations are identical, while operators acting on the driving potentials \( W_{21}, \tilde{W}_{21} \) are not. Assuming harmonic time dependence \( \exp(i\omega t) \) and \( W_{21} = \tilde{W}_{21} = 0 \) we obtain the fourth-order polynomial equations for the eigenfrequencies (roots). The four roots can be found using a perturbation method and turn out to be \( \sigma_{e_{W}}, \sigma_{e_{W}}^*, \sigma_{nd} \) and \( -\sigma_{nd}^* \), where * denotes the complex conjugate and two of these quantities are given in (5).

Having found the eigenfrequencies, the response of the system can be expressed in the frequency domain. By partial fraction expansion, the contributions of the different roots can be separated. The response functions (6) and (8) take only the contribution of the root \( \sigma_{nd} \) into account. If we add the contribution of \( \sigma_{nd}^* \), we obtain an expression which corresponds to that of a harmonic oscillator:

\[
F_{\text{NDFW}}(\sigma) = \frac{\tilde{A}'}{(\sigma_{\text{real}}^2 + \sigma_{\text{imag}}^2) - \sigma^2 + 2i\sigma\sigma_{\text{imag}}}.
\]

where \( \sigma_{nd} = -\sigma_{\text{real}} - i\sigma_{\text{imag}} \) [see (5)]. The contribution from the root \( -\sigma_{nd}^* \) is added only to establish the equivalence with the harmonic oscillator; its effect on the results is negligible as is the contribution from the Chandler wobble. Hence it is convenient to compare the resonant admittances \( R \) with that of a damped harmonic oscillator driven at frequency \( \sigma \):

\[
F_{\text{HO}}(\sigma) = \frac{\tilde{A}}{\sigma_{\text{real}}^2 - \sigma^2 + i2\gamma\sigma},
\]

where in our case \( \sigma_{\text{real}}^2 = \sigma_{\text{real}}^2 + \sigma_{\text{imag}}^2 \) and \( \gamma = \sigma_0/Q = \sigma_{\text{imag}} \). In the case of the harmonic oscillator \( \sigma_0 \) is the eigenfrequency of the undamped system and \( \gamma, Q \) are the damping and quality factors, respectively.

In order to remove the frequency independent contribution from the resonant admittance, we had subtracted the admittance at \( \sigma_{O} \) from those at other tidal frequencies. Therefore, we take into account a remaining small resonance contribution at frequency \( \sigma_{O} \), and introduce in (11) a corresponding term, so that the complete model function represents the difference of the frequency response at \( \sigma_{j} \) from the same quantity at reference frequency \( \sigma_{O} \). That leads to

\[
F_{\text{M}}(\sigma) = \frac{\tilde{A}}{\sigma_{\text{real}}^2 - \sigma_j^2 + i2\gamma\sigma_j} - \frac{\tilde{A}}{\sigma_{\text{real}}^2 - \sigma_{O}^2 + i2\gamma\sigma_{O}},
\]

where \( \sigma_j \) are the tidal frequencies of \( P_1, K_1, \psi_1 \) and \( \phi_1 \). \( \tilde{A} \) is the strength of resonance which we take to be complex. That allows for a phase difference between the forcing function and the Earth's response, additional to the one caused by resonance. In the case of gravity, the imaginary part of \( \tilde{A} \) should be small [see (8)]. For an anelastic earth a slight frequency-dependence of \( \tilde{A} \) is expected which we neglect due to the narrow frequency range we are dealing with here.

The resonance description by Wahr (1981) is equivalent with a near-resonance approximation of the harmonic oscillator (Zürn et al. 1986). We prefer to use the analytical model described in Section 2 because it provides the physical meaning for \( \tilde{A} \) and \( \sigma_{nd} \).

5 Stacking method

We fit the model function (11) to the so-called resonant admittances \( R \). Since (11) is not
linear in the parameters $\tilde{A}$, $\sigma_0$, $\gamma$, to be determined, we use the Marquardt algorithm (Marquardt 1963) in a linearized least-squares estimation for a complex function.

The well-known formulation for the residual error $s$, which has to be minimized iteratively, is

$$s^2 = \sum_j W_j (F_{Mj} - R_j)^2.$$  \hspace{1cm} (12)

The subscript $j$ denotes the tidal frequencies, $F_{Mj}$ and $R_j$ are the model function and resonant admittances, respectively. $W_j = 1/\epsilon_j^2$ are the weighting factors, where $\epsilon_j$ are the standard deviations of the tidal admittances (Table 2).

In a first analysis we fitted the resonant admittances for each tidal station separately, and later refer to these as the individual fits.

The basic idea of the next two steps is to consider the resonant admittances for different stations simultaneously. Since we are searching for global parameters $\sigma_0$, $\gamma$, any tidal instrument at any tidal station should provide the same values for these parameters. Therefore, in a second analysis we fit the same model function (11) to the resonant admittances of all stations simultaneously. We refer to that procedure as the stacking method no. 1. The subscript $j$ in (12) now runs over four tidal frequencies at each of six stations (24 complex resonant admittances, four unknowns). The resulting strength of resonance $\tilde{A}$ must then be interpreted as a weighted average of the values for each station.

In a third analysis – stacking method no. 2 – we take into account that the strength of the resonance can differ for each tidal record (e.g. due to different calibration, latitude, local effects), while the values for $\sigma_0$ and $\gamma$ should remain the same. That is most certainly the case when different tidal components (tilt, gravity, strain) are used for an equivalent investigation (see Neuberg & Zurn (1986) for the case of cavity effects). Since the resonant part of all tidal admittances can be expressed by a linear Love number combination (Wahr 1981), the frequency dependence remains the same, while the strengths of resonance differ according to different combinations of Love numbers. In the case of the second stacking method, (12) is modified to

$$s^2 = \sum_k \sum_j W_{kj} [F_{Mjk} (\sigma_j, \tilde{A}_k, \sigma_0, \gamma) - R_{jk}]^2, \hspace{1cm} (13)$$

where $k$ denotes the different tidal stations. Note that the common parameters $\gamma$ and $\sigma_0$ have no subscript. Since the mentioned weighting factors have the second subscript $k$ as well, the different signal-noise ratios for the tidal stations are taken into account automatically. The weighted average $\tilde{A}$ (stacking method no. 1) is now replaced by individual values $\tilde{A}_k$ for each tidal station. This means an improvement of the estimation, which on the other hand reduces the number of degrees of freedom in the fit. In the case of six time series we have 14 unknowns in comparison with only four unknowns, using the first stacking method.

6 Uncertainties due to ocean loading

In addition to the errors in the resulting parameters $\gamma$ and $\sigma_0$ which are obtained by considering the curvature matrix of the least-squares estimation, systematic errors are superimposed due to uncertainty in the ocean load-corrections. These errors, which can hardly be assessed in detail, enter the analysis via data reduction and contaminate the results. By the following procedure we investigate their possible influence on the resulting resonance parameters.

We chose a kind of Monte-Carlo method and varied the given correction values (Ducarme, private communication), considered in the complex plane, randomly (uniformly distributed) within a certain area. The size of this area is chosen to be $\pm 40$ per cent of the given correc-
PERIOD OF THE ASSOCIATED NUTATION [SDAYS]

<table>
<thead>
<tr>
<th>Period</th>
<th>Eigenfrequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.0133</td>
<td>434.2</td>
</tr>
<tr>
<td>15.0130</td>
<td>434.6</td>
</tr>
</tbody>
</table>

Figure 1. Results from Table 4 in the $Q-\alpha_0$-plane. Stacking no. 2: full circle, stacking no. 1: open triangle, stacking no. 2 with superconducting meters only: open circle. The errorbars represent one standard deviation, the crosses are results of the Monte-Carlo method as described in Section 6. Inset sketches the corresponding variation of ocean load-corrections. Eigenfrequency from theory (Sasao et al. 1980) and VLBI observations (Gwinn et al. 1986) are indicated on $\sigma_0$-axis.
tion value, which is located in the centre of the area (see inset in Fig. 1). The amount of 40 per cent is based on a 'pessimistic assessment' by Baker (private communication) and Woodworth (1985). All relevant ocean correction values were treated this way. For each of 200 combinations of randomly varied correction values, we fitted the model function to these corrected resonant admittances using stacking method no. 2. The variation of correction values causes of course a corresponding spreading of the resulting resonance parameters, as illustrated in the parameter plane for $Q$ and $\sigma_0$ (Fig. 1).

7 Results and discussion

The resulting resonance parameters calculated from the individual fits are listed in Table 3. We present these values for completeness rather than to draw conclusions from them. The values derived from Berlin, Potsdam and Strasbourg data are obviously unreasonable (negative $Q$), if considered separately.

For interpretation we are only concerned in those results (Table 4) which have been deduced by using the stacking methods. In addition to the attempts described above, the stacking method no. 2 was also applied to data based on recordings from superconducting gravimeters only. The values for the complex strength of resonance $\tilde{A}$ consist mainly of a real part as expected. A corresponding plot is shown in Fig. 2. The results for the quality factor and the eigenfrequency are plotted in the $Q$, $\sigma_0$ parameter plane in Fig. 1.

The imaginary parts of the estimated strengths $\tilde{A}_k$ are about 3 per cent of the real parts. The strength $\tilde{A}_k$ appears as a real quantity in (8) if the Love numbers involved are real quantities. One possible cause of the observed imaginary parts could be mantle anelasticity, because it would make these Love numbers complex with a small imaginary part (e.g. Zschau 1978). Another cause could be a known defect in the ocean load-corrections: the implicit contribution of the core resonance in Schwiderski's ocean tide-model, especially $K_1$ (Wahr 1983). We point out again that for diurnal gravity tides in Central Europe this effect should be small because total ocean contributions are very small. Basically the complex strengths (as a function of the observed signal) can be important sources of information on Earth's elastic and anelastic characteristics (e.g. internal pressure 'Love numbers', see Yoder & Ivins 1987).

Table 3. Resonance parameters for each tidal station calculated from the individual fits.

<table>
<thead>
<tr>
<th>Station</th>
<th>$Q$</th>
<th>$\sigma_0$</th>
<th>FCN-eigenperiod</th>
<th>$\tilde{A}_k$ Real</th>
<th>$\tilde{A}_k$ Imag</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>[deg/h]</td>
<td>[sidereal days]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Berlin</td>
<td>-0.79</td>
<td>15.0467 ± 0.002</td>
<td>2629 ± 15372</td>
<td>0.00001 ± 0.00059</td>
<td>0.000059</td>
</tr>
<tr>
<td>Potsdam</td>
<td>-1.332</td>
<td>15.0757 ± 0.0032</td>
<td>435 ± 40</td>
<td>-0.00052 ± 0.00005</td>
<td>0.00014 ± 0.00005</td>
</tr>
<tr>
<td>Bruxelles</td>
<td>2.035</td>
<td>15.0760 ± 0.0009</td>
<td>430 ± 12</td>
<td>-0.00065 ± 0.00002</td>
<td>-0.00002 ± 0.00002</td>
</tr>
<tr>
<td>Frankfurt</td>
<td>0.134</td>
<td>15.0759 ± 0.0006</td>
<td>431 ± 8</td>
<td>-0.00061 ± 0.00001</td>
<td>-0.00006 ± 0.00001</td>
</tr>
<tr>
<td>Schiltach</td>
<td>0.1076</td>
<td>15.0766 ± 0.0040</td>
<td>423 ± 48</td>
<td>-0.00067 ± 0.00008</td>
<td>-0.00009 ± 0.00008</td>
</tr>
<tr>
<td>Strasbourg</td>
<td>-0.2158</td>
<td>15.0636 ± 0.0069</td>
<td>666 ± 210</td>
<td>-0.00038 ± 0.00012</td>
<td>0.00007 ± 0.00012</td>
</tr>
</tbody>
</table>
Table 4. Resonance parameters from stacking method no. 1 and no. 2 applied to all data and additionally stacking method no. 2 for the superconducting gravimeters in Bruxelles and Frankfurt only.

<table>
<thead>
<tr>
<th>Stacking #1:</th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td></td>
<td>deg/h</td>
</tr>
<tr>
<td>2758 ± 536</td>
<td>15.0756 ± 0.0005</td>
<td>435 ± 7</td>
</tr>
</tbody>
</table>

| Stacking #2 |
|--------------|---------------|
| Berlin       | -0.00069 ± 0.00006 |
|              | 0.00003 ± 0.00006 |
| Potsdam      | -0.00053 ± 0.00001 |
|              | -0.00001 ± 0.00001 |
| Bruxelles    | -0.00064 ± 0.00001 |
| Frankfurt    | -0.00001 ± 0.00001 |
|              | -0.00001 ± 0.00001 |
| Schiltach    | -0.00065 ± 0.00002 |
|              | -0.00001 ± 0.00002 |
| Strasbourg   | -0.00060 ± 0.00002 |
|              | -0.00003 ± 0.00002 |
| Bruxelles    | -0.00065 ± 0.00001 |
|              | -0.00001 ± 0.00001 |
| Frankfurt    | -0.00001 ± 0.00001 |
|              | -0.00001 ± 0.00001 |

Our Q-estimate of about 2800 (corresponding to a decay time \( \tau = 2Q/\sigma_0 = \text{Im}^{-1}(\sigma_0) = 2.4 \text{yr} \)) differs appreciably from the value of 840 found by Goodkind (1983). We believe our results to be more reliable because of the smaller ocean corrections in our case and the advantages of stacking methods. Very recent determinations of the decay time of the FCN from VLBI-observations by Eubanks et al. (1986) and Herring et al. (1986) are about 19.

**Figure 2.** Strength of resonance in the complex plane. Stacking no. 2: full circle (the capital letters denote tidal stations), stacking no. 1: open triangle, stacking no. 2 with superconducting meters only; open circle. Errorbars represent one standard deviation.
years ± 50 per cent. Obviously this modal $Q$ cannot depend on the kind and place of observation. In view of the large error bars on all these determinations, no conclusions should be drawn concerning their possible origin. Clearly a combination of physical effects acts to damp the eigenmode: mantle anelasticity, visco-magnetic friction and topographic coupling at the CMB and friction in the oceans. Mantle anelasticity alone cannot explain our decay times. Wahr & Bergen (1986) found decay times between 55 and 340 yr for different anelastic models of the mantle.

In spite of the spreading due to the described ocean load treatment (see Section 6) the eigenfrequency $\omega_0$ differs significantly from the theoretical values by Wahr (1981) and Sasao, Okubo & Saito (1980). However, this result is in good agreement with the two recent determinations from VLBI data by Eubanks et al. (1986) and Herring et al. (1986). It is a well-known fact that the influence of the ocean on the Chandler wobble is to lengthen the period (Smith & Dahlen 1981) by about 30 days. We can expect a similar but probably much smaller effect on the period of NDFW (Legros & Amalvic 1985). Whatever the size of the effect is, the shift would be in the wrong direction. This is also the case for the effect of dispersal due to mantle anelasticity, on the NDFW-period (Wahr & Bergen 1986; Dehant 1986). Relaxation of the shear moduli in the mantle occurs for all reasonable mantle rheologies (e.g. Anderson & Minster 1979).

By examination of (5) we find that the dynamical ellipticity of the core $\varepsilon^L$ and the elastic parameter $h^L_j$ (proportional to the internal pressure ‘Love number’) have the most effect on the eigenfrequency. Visco-magnetic coupling parameters $K'$ and $K$ are too small by orders of magnitude (Hinderer et al. 1986; Sasao et al. 1980). The observed shift in $\omega_0$ would imply an increase in core flattening by about 6 per cent as discussed by Gwinn, Herring & Shapiro (1986) which corresponds to a deviation from the hydrostatic figure of the CMB of about 500 m (for a zonal spherical harmonic of order 2). Large amplitude heterogeneities of shorter wavelengths in the outermost core and topography on the CMB are recently suggested by mantle convection and seismological investigations (Hager et al. 1985; Creager & Jordan 1986; Morelli & Dziewonski 1987; Poupinet, Pilet & Souriau 1983).

The second term in (5) shows that a decrease of CMB deformation (due to a stiffer lowermost mantle) caused by the fluid pressure associated with the wobble provides another possibility for the frequency shift. The observation thus presents a new constraint for models of the Earth in the vicinity of the core-mantle boundary.

8 Acknowledgments

We sincerely thank the following colleagues for providing data and preprints: B. Ducarme (Bruxelles), who also provided ocean load calculations, G. Jentzsch (Berlin), R. Lecolazet (Strasbourg) and B. Richter (Frankfurt). Discussions with T. Baker (Bidston), D. C. Agnew and J. Goodkind (La Jolla), P. A. Rydelek (Los Angeles), H. Wilhelm (Karlsruhe) and H. Legros (Strasbourg) are gratefully acknowledged. We also thank G. Bartman who proficiently typed the manuscript and U. Zimmermann who drew the figures and helped with the calculations. Financial support was provided by the DFG (Deutsche Forschungsgemeinschaft) under grant number Wi 687/1-1 and INSU (Institut National des Sciences de l'Univers, France) grant number ATP-Noyau 16-14. The computations were carried out at the Computer Centre of Karlsruhe University.

References