Propagation invariants, reflection and transmission in anisotropic, laterally heterogeneous media

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SUMMARY
For quasi-stratified media in which the principal variation in seismic properties is with depth, propagation invariants can be constructed from certain combinations of the displacement and tractions elements of two seismic wavefields. These invariants are independent of depth and vanish for identical wavefields, and are constructed for anisotropic, laterally varying media in the spatial and wavenumber domains.

These propagation invariants can be exploited to substantially simplify the construction of reflection and transmission processes in laterally varying media, including coupling between wavenumbers. The implementation of this approach is illustrated by application to the incidence of SH-waves on an irregular interface below a free surface. The results are in excellent agreement with those from other schemes but take about 20 per cent less computation time. Even greater improvements in calculation speed are possible in more complex models.

Key words: anisotropy, lateral heterogeneity, reflection, seismic waves, transmission.

1 INTRODUCTION
The calculation of theoretical seismograms for laterally varying media by wavenumber coupling techniques, in the frequency–wavenumber domain, has been shown to be feasible but computationally intensive (see e.g. Koketsu 1987; Haines 1988). It is therefore desirable to be able to reduce the computational demands in such situations. We show here how this can be achieved by exploiting the use of propagation invariants of the seismic wavefield in laterally varying media.

In a horizontally stratified medium, propagation invariants of the seismic wavefield built up from two different solutions of the seismic wave equations, have been used to establish symmetry and unitarity properties of reflection and transmission coefficients (Kennett, Kerry & Woodhouse 1978). Such invariants have also been shown to play an important role in the representation of the response of a medium to excitation by a source (Kennett 1983, chapter 7).

Haines (1988) has been able to extend the concept of such invariants to acoustic waves propagating in a laterally varying medium by working with a representation of the wavefield in the spatial wavenumber domain. In this paper we show how Haines' work can be extended to anisotropic, laterally inhomogeneous media and are also able to provide a link to a spatial propagation invariant (Kennett 1984).

The propagation invariants depend on the interaction of two distinct wavefields and vanish if both fields are identical. This result can be exploited to simplify the calculation of reflection and transmission problems by selective extraction of appropriate parts of the wavefield. Such a simplification is illustrated for the transmission of SH-waves through an irregular interface.

2 PROPAGATION INVARIANTS
2.1 A spatial invariant
We will first establish a general propagation invariant for time harmonic elastic fields (Kennett 1984) closely related to Betti's theorem, which will be very useful for quasi-stratified media where the predominant variation in the elastic parameters is in the z-direction. We will assume a time dependence \( \exp(-i\omega t) \), but will not normally represent the time variation explicitly.

Consider two elastic displacement fields \( u_1, u_2 \) with associated stress-tensor fields \( \tau_1, \tau_2 \). In Cartesian coordinates, the equations of motion in the absence of sources are

\[
\partial_t \tau_{1ij} = -\rho \omega^2 u_{1i}, \quad \partial_t \tau_{2ij} = -\rho \omega^2 u_{2i},
\]

(2.1)
where
\[ \tau_{ij} = c_{ijkl} \partial_{k} U_{l}, \tag{2.2} \]
and the elastic modulus tensor \( c \) has the symmetries
\[ c_{ijkl} = c_{jikl} = c_{ijlk} = c_{iklj}. \tag{2.3} \]
We form the scalar product of each equation of motion with the alternate displacement field and then subtract to eliminate the frequency dependent term. Then, on integrating over a volume \( V \) in which stresses and displacements are continuous
\[ \int_{V} dV(\partial_{j} \tau_{ij} - \partial_{j} \tau_{2ij} U_{1l}) = 0. \tag{2.4} \]
Now, using the divergence theorem, we find that over the bounding surface \( S \) of \( V \)
\[ \int_{S} n_{j}(\partial_{2j} \tau_{1ij} - \partial_{2j} \tau_{2ij} U_{1l}) = 0, \tag{2.5} \]
since the other terms cancel because of the linear stress-strain relation (2.2). In terms of the traction vector \( t \) for the surface \( S \) we can rewrite (2.5) in the more compact form
\[ \int_{S} dS(\partial_{2j} t_{1ij} - \partial_{2j} t_{2ij} U_{1l}) = 0, \tag{2.6} \]
where \( t_{j} = n_{j} \tau_{ij}. \)

Although (2.6) has been derived under the assumption of continuous stress and displacement fields, the result can be extended to include regions containing material discontinuities. Under conditions of welded contact, both displacement and traction will be continuous across such a discontinuity, and in the application of the divergence theorem the additional integrals introduced on the two sides of the discontinuity will cancel.

Consider now a quasi-stratified medium for which the main variation in elastic properties is in the \( z \)-direction (Fig. 1). We construct surfaces \( S_{1} \) and \( S_{2} \) spanning the whole \( x-y \) plane, which could follow material discontinuities or contours in elastic wavespeeds depending on the nature of the model. If we now choose the surface \( S \) in (2.6) to include both \( S_{1} \) and \( S_{2} \) with completion surfaces \( S_{m} \) at infinity, then
\[ \int_{S_{m}} dS(\partial_{2j} t_{1ij} - \partial_{2j} t_{2ij} U_{1l}) = \int_{S_{1}} dS(\partial_{2j} t_{1ij} - \partial_{2j} t_{2ij} U_{1l}), \tag{2.7} \]
where we have taken the normals to \( S_{1} \) and \( S_{2} \) to lie in the positive \( z \)-direction. This result requires \( u_{1}, u_{2} \) to decay sufficiently rapidly as \( S_{m} \) is moved to infinity, which can be assured for a quasi-stratified medium with all source confined to finite \((x, y)\) values.

We have therefore established a spatial propagator invariant for an arbitrary anisotropic medium. The integral
\[ H(u_{1}, u_{2}; S) = \int_{S} dS(u_{2j} t_{1ij} - t_{2j} U_{1l}), \tag{2.8} \]
will be invariant for surfaces \( S \) spanning the \( x-y \) plane at different levels. We note that (2.8) will vanish identically if \( u_{1} \) and \( u_{2} \) are the same field. As a result any part of \( u_{1} \), which is a multiple of \( u_{1} \) makes no contribution to the invariant (2.8), i.e. if
\[ u_{2} = cu_{1} + u_{2}^{*} \]
for some constant \( c \), then only the part involving \( u_{1}^{*} \) will give a non-zero contribution to (2.8). With a suitable choice of \( u_{1} \), we can therefore extract appropriate parts of the field from \( u_{2} \).

### 2.2 Invariants in the coupled wavenumber domain

For a quasi-stratified medium we can express the effects of the variation of elastic parameters with depth \( z(x, y) \) by a set of ordinary differential equations in the spatial Fourier domain. The influence of horizontal variability is then included by coupling between different horizontal wave number components (Kennett 1972, 1986).

For a general anisotropic medium, we introduce the set of elastic modulus matrices \( C_{ij} \) (Woodhouse 1974) such that
\[ (C_{ji})_{kl} = c_{iklj}, \tag{2.9} \]
and the traction vectors \( \tau_{l} \) corresponding to normals along the coordinate axes,
\[ (\tau_{l})_{ij} = \tau_{ij}. \tag{2.10} \]

In terms of the displacement vector \( u \),
\[ \tau_{l} = C_{ij} \partial_{i} u + C_{ij} \partial_{j} u, \quad o = 1, 2. \tag{2.11} \]
We will adopt the convention that Greek suffices will always refer to summation over horizontal coordinates.

The equation of motion (2.1) can be recast as
\[ \partial_{i} \tau_{ij} = -\rho \omega^{2} u. \tag{2.12} \]

On rearranging (2.11), (2.12) so that derivatives with respect to the \( x_{3} \) coordinate \((z)\) appear only on the left-hand side of the equations we obtain
\[ \partial_{3} u_{ij} = -C_{33} C_{30} \partial_{3} u + C_{33} \partial_{j}, \tag{2.13} \]
\[ \partial_{3} \cdot \tau_{3} = -\rho \omega^{2} \cdot u - \partial_{3}(Q_{\nu \nu} \partial_{j} u) - \partial_{3}(C_{03} C_{33} \partial_{3} \tau_{3}), \tag{2.14} \]
where the matrix \( Q_{\nu \nu} \) is given by
\[ Q_{\nu \nu} = C_{\nu \nu} - C_{\nu 3} C_{33} C_{33} \tau_{3}. \tag{2.15} \]

A more detailed derivation of (2.13, 2.14) can be found in Kennett (1986).

In order to simplify subsequent notation we write
\[ J_{03} = C_{33}^{-1} C_{30}, \quad K_{33} = C_{33}^{-1}, \tag{2.16} \]
and as a result of the symmetry of the elastic modulus tensor
\[ J_{33}^T = C_{33} C_{33}^{-1}, \]
so that (2.13), (2.14) depend on the elastic moduli through the expressions \( Q_{\nu \nu}, J_{33}, K_{33}. \)

We now take the Fourier transform of equations (2.13), (2.14) over the horizontal coordinates \( x_1, x_2. \) We set
\[ F[g(x)] = \hat{g}(k, x_3) \]
\[ = \int dx_1 \int dx_2 g(x_1, x_2, x_3) \exp \left[ -i(k_1 x_1 + k_2 x_2) \right], \]
and use the result that the Fourier transform of a product is the convolution of the Fourier transforms of the individual terms. The resulting differential-integral equations are conveniently written in matrix form
\[ \frac{\partial}{\partial x_3} \left[ \begin{array}{c} \hat{u}(k, x_3) \\ \hat{e}(k, x_3) \end{array} \right] = \mathbf{J} \hat{e}(k, x_3), \]
where \( \mathbf{J} \) is a 3 \( \times \) 3 unit matrix. In terms of the displacement-traction vector
\[ \mathbf{b}(k, x_3) = [\hat{u}(k, x_3), \hat{e}(k, x_3)]^T, \]
we can rewrite (2.19) in the compact form
\[ \frac{\partial}{\partial x_3} \mathbf{b}(k, x_3) = \int d\xi \mathbf{A}(k, \xi, x_3) \mathbf{b}(\xi, x_3), \]
where the effect of lateral variations are contained in the matrix \( \mathbf{A}, \) in Fourier transform elements of the form
\[ \mathbf{J} \hat{e}(k - \xi). \]

Haines (1988) derived a similar equation to (2.19) for the propagation of acoustic waves and pointed out that the same Fourier transform elements would occur in \( \mathbf{A} \) if we make the substitutions
\[ k \rightarrow -k, \quad -k \rightarrow -\xi. \]

We may exploit this property of \( \mathbf{A} \) to build up propagation invariants for the seismic wavefield in a laterally heterogeneous, anisotropic medium. Consider the derivative of \( \langle \mathbf{b}_1, \mathbf{b}_2 \rangle \) with respect to \( x_3: \)
\[ \partial_3 \langle \mathbf{b}_1, \mathbf{b}_2 \rangle = \int d\xi \hat{e}(k, x_3)^T \mathbf{A}(k, \xi, x_3) \mathbf{b}(\xi, x_3), \]
where for notational simplicity we have written \( t \) for \( z_3. \) Our previous spatial invariant (2.8) taken over a plane \( x_3 = \text{const} \) would be
\[ H(\mathbf{u}_1, \mathbf{u}_2; x_3) = \int dx_1 \int dx_2 \left[ \mathbf{u}_1^T(x) \mathbf{e}_T(x) - \mathbf{e}_T(x) \mathbf{u}_2(x) \right]. \]

The integral over the \( x_1-x_2 \) plane is equivalent to the Fourier transform evaluated at zero argument, which for a product has the form
\[ F[uv](0) = \int dk u(-k) v(k), \]
and so we can recognize the equivalence of the representations (2.32) and (2.33) for the propagation invariant. The wavenumber domain form (2.32) is particularly useful in the description of the reflection and transmission of seismic wavefields.

The idea of a propagation invariant can be extended to multiple solutions of the seismic wave equations. We introduce a 3 \( \times \) 6 displacement-traction matrix \( \mathbf{B}(k, x_3) \) whose columns are three linearly independent vector solutions of the differential-integral equations for the seismic
field (2.21). Such a field matrix can for example correspond to the incidence of three different wavetypes at the top of a region and will itself satisfy the differentio-integral equation

\[ \hat{z}_iB(k, x_3) = \int d\xi A(k, \xi, x_3)B(\xi, x_3). \]  

(2.35)

We define the matrix invariant for two field matrices \( B_1 \) and \( B_2 \) as

\[ (B_1, B_2) = \int dk B_1^T(-k, x_3)NB_2(k, x_3), \]

(2.36)

so that the \( j \)th entry of \( (B_1, B_2) \) is the expression \( \langle b_{1j}, b_{2j} \rangle \) constructed from the \( i \)th column of \( B_1 \) and the \( j \)th column of \( B_2 \). From the definition (2.32)

\[ (B_1, B_2)^T = - (B_2, B_1). \]

(2.37)

When the three columns of \( B_1 \) satisfy a common boundary condition at some level, such as a radiation condition, \( (B_1, B_2) = 0 \).

(2.38)

It is interesting to note that for a horizontally stratified medium the property (2.28) for the matrix \( A \) with \( \xi = k \) generates the symmetry property of the first-order differential equations that Thomson, Clarke & Garmany (1986) were unable to associate with a propagation invariant. In such a stratified medium the components in horizontal wavenumber propagate independently and so the integral over \( k \) can be dropped from (2.29). Thus, in the special case of a horizontally stratified medium

\[ b_i^T(-k, x_3)NB_i(k, x_3), \]

(2.39)

will be a propagation invariant for each \( k \).

### 3 Reflection and Transmission Problems

In this section we show how propagation invariants can be used to simplify the calculation of reflection and transmission problems in laterally varying media, including coupling between wavenumbers. We will first show how formal results for reflection and transmission matrices can be constructed in terms of propagation invariants for a zone of heterogeneity bordered by uniform half-spaces. Then we indicate how such results can be applied directly in the case of an irregular interface between two uniform media.

#### 3.1 The use of propagation invariants

In order to give a clear physical interpretation to the concepts of up and downgoing waves, and hence to reflection and transmission problems, we will consider a laterally heterogeneous region in \( (z_A, z_B) \) bordered by uniform half-spaces (Fig. 2). Such an ‘invariant embedding’ procedure allows us to make an unambiguous decomposition of the seismic wavefield into up and downgoing components within the two half-spaces \( (z < z_A, z > z_B) \).

For the half-space \( z < z_A \) we introduce the \( 3 \times 6 \) field matrices \( B_{UA}(k, z), B_{DA}(k, z) \) corresponding to the displacement and traction elements for upward or downward propagating waves in each of the three wavetypes (e.g. \( P, SV \) and \( SH \) for an isotropic medium). Any

displacement-traction field \( b(k, z) \) (a 6-vector) in \( z < z_A \) can then be constructed as a linear combination of these two field matrices:

\[ b(k, z) = B_{UA}(k, z)u_{UA}(k) + B_{DA}(k, z)u_{DA}(k), \]

(3.1)

where \( u_{UA}(k), u_{DA}(k) \) are the 3-vectors of the weighting factors for the up and downgoing wave elements. If we take \( z_A \) as the reference level for phase, \( B_{UA}(k, z_A) \) and \( B_{DA}(k, z_A) \) represent the appropriate columns of the eigenvector matrix \( D_A \) for the uniform medium at wavenumber \( k \) (see e.g. Kennett 1983, chapter 3).

In a similar way, for the lower half-space \( z > z_B \) we can introduce the field matrices \( B_{UB}(k, z), B_{DB}(k, z) \) and represent the seismic wavefield in \( z > z_B \) as

\[ b(k, z) = B_{UB}(k, z)u_{UB}(k) + B_{DB}(k, z)u_{DB}(k). \]

(3.2)

The field matrices \( B_{UA}, B_{DA}, B_{UB}, \) and \( B_{DB} \) have been introduced for the uniform half-spaces bordering the zone of heterogeneity \( (z_A, z_B) \). The definition of each of the matrices can, in principle, be extended into the heterogeneous zone by solving the differentio-integral equations (2.35) with the boundary condition of equality to the uniform medium form at \( z_A \) or \( z_B \), as appropriate. Within the zone of heterogeneity the coupling between horizontal wavenumbers will destroy any direct interpretation of the field matrices \( B_i \) in terms of up and downgoing waves. However, the representations (3.1) and (3.2) for the stress and displacement field can still be employed for \( z \) in \( (z_A, z_B) \) with the new forms for \( B_i \).

If we now wish to represent the result of downgoing incident waves on the laterally heterogeneous region \( (z_A, z_B) \) we have to connect the fields on the two sides of the region. We require there to be no upgoing waves in \( z > z_B \) and so \( u_{UB}(k) = 0 \). We can then equate the two forms (3.1) and (3.2) for the seismic wavefield at some level \( z \), but we still need to connect \( u_{UA}(k) \) and \( u_{DB}(k) \) to the downgoing wavector \( u_{DA}(k) \). We can achieve this by a generalization of the concept of reflection and transmission coefficients.

We follow Kennett (1986) and introduce reflection and transmission matrices for incident downward waves at \( z_A \), \( R^{DA}(k, k_0) \) and \( T^{DA}(k, k_0) \) which represent the fields transferred to wavenumber \( k \) from incident wavenumber \( k_0 \). In this way we can describe the upgoing field in \( z < z_A \) via

\[ u_{UA}(k) = \int d\xi R^{DA}(k, \xi)u_{DA}(\xi), \]

(3.3)
and the downgoing field in \( z > z_B \)
\[
\psi_{DB}(k) = \int d\xi \hat{F}_\xi(k, \xi) \psi_{DA}(\xi).
\]

(3.4)

The requirement that the forms of the wavefield in the upper and lower half-spaces should match leads to
\[
B_{DA}(k, z) \psi_{DA}(k) + B_{UA}(k, z) \int d\xi \hat{F}_\xi(k, \xi) \psi_{DA}(\xi)
= B_{DB}(k, z) \int d\xi \hat{F}_\xi(k, \xi) \psi_{DA}(\xi).
\]

(3.5)

Because of the coupling between wavenumbers introduced by the lateral heterogeneity we have to consider all the wavenumbers together. It is therefore computationally and notationally convenient to discretize the wavenumber domain. We introduce the ‘super-vector’ \( \psi_{DA} \) of downgoing wave elements and the field ‘super-matrices’ such as \( B_{DA}(z) \) whose entries for wavenumber \( k \) are just \( \psi_{DA}(k) \) and \( B_{DA}(k, z) \). We can then express (3.5) as
\[
B_{DA}(z) \psi_{DA} + B_{UA}(z) R_{DA}^B \psi_{DA} = B_{DB}(z) T_{DB}^A \psi_{DA}.
\]

(3.6)

in terms of reflection and transmission ‘super-matrices’ \( R_{DA}^B \) and \( T_{DB}^A \) allowing for full interconversion between wavenumbers. Since (3.6) must be valid for an arbitrary downgoing wavefield \( \psi_{DA} \), we require
\[
B_{DA}(z) + B_{UA}(z) R_{DA}^B = B_{DB}(z) T_{DB}^A,
\]

(3.7)

which represent two distinct sets of equations for displacements and tractions. Conventional methods of solution for \( R_{DA}^B \) would require the elimination of \( T_{DB}^A \) between the displacement and traction equations, which requires at least five inversions of large matrices. The process is substantially simplified by the use of propagation invariants.

We recall that the construction of the propagation invariants (2.36) for the field matrices \( B_i(k, z) \) requires integration over all wavenumbers, which reduces to summation over all wavenumbers in the discrete domain. This procedure can be regarded as a direct operation on the equivalent super-matrices \( B_i(z) \). We can thus identify the invariant for two super-matrices \( B_i \), \( B_j \) with the invariant over all wavenumbers for the corresponding field matrices \( B_1 \), \( B_2 \):
\[
\langle B_1, B_2 \rangle = \langle B_1 \rangle \langle B_2 \rangle = \sum_k B_1(ke^{-1}) N B_2(k).
\]

(3.8)

We recall
\[
\langle B_1, B_1 \rangle = 0,
\]

(3.9)

so that by suitable choice of \( B \) field in forming an invariant with both sides of equation (3.7), either \( R_{DA}^B \) or \( T_{DB}^A \) can be readily extracted.

First, forming the invariant with \( B_{UA} \) we remove the term in \( R_{DA}^B \),
\[
\langle B_{UA}, B_{DA} \rangle = \langle B_{UA}, B_{DB} \rangle T_{DB}^A,
\]

(3.10)

and so
\[
T_{DB}^A = \langle B_{UA}, B_{DB} \rangle^{-1} \langle B_{UA}, B_{DA} \rangle.
\]

(3.11)

Second, by making an invariant with \( B_{DB} \) we leave only the reflection super-matrix
\[
\langle B_{DB}, B_{DA} \rangle + \langle B_{DB}, B_{UA} \rangle R_{DB}^A = 0,
\]

(3.12)

and hence
\[
R_{DB}^A = -\langle B_{DB}, B_{UA} \rangle^{-1} \langle B_{DB}, B_{DA} \rangle.
\]

(3.13)

However as noted above (2.36),
\[
\langle B_{DB}, B_{UA} \rangle = -\langle B_{UA}, B_{DB} \rangle^T,
\]

(3.14)

and so only a single matrix need be inverted to generate both the reflection and transmission super-matrices.

A similar development may be made for upgoing waves incident at \( z_B \) to calculate the super-matrices \( R_{UB}^A \), \( T_{UB}^A \).

The construction procedure for the reflection and transmission super-matrices is formally analogous to the development for a single wavenumber in a horizontally stratified medium presented in Section 5.2 of Kennett (1983). In that case the invariant between up and downgoing wave solutions takes a particular simple form with consequent simplifications of the expressions in transmission. The formal equivalence enables us to establish immediately a set of symmetry relations for the reflection and transmission super-matrices for an anisotropic, laterally varying medium:
\[
R_{DB}^A = (R_{DA}^B)^T,
\]

(3.15)

\[
R_{DB}^C = (R_{DB}^A)^T,\]

(3.16)

\[
T_{DB}^A = (T_{DB}^A)^T.
\]

These relations reduce for acoustic waves to the results deduced by Haines (1988) with a somewhat different definition for the reflection and transmission super-matrices.

The direct analogy with the stratified medium results (but with substantially higher dimensionality) means that we can find the reflection and transmission super-matrices for a region \((z_A, z_B)\) in terms of the matrices for the subregions \((z_A, z_C)\) and \((z_B, z_C)\), e.g.
\[
R_{DA}^C = R_{DB}^A + T_{DA}^B R_{DB}^C \langle 1 - R_{DA}^B R_{DB}^C \rangle^{-1} T_{CB}^D.
\]

(3.17)

\[
T_{DA}^C = T_{DB}^A \langle 1 - R_{DA}^B R_{DB}^C \rangle^{-1} T_{CB}^B.
\]

(3.18)

Such results can be very useful when combining laterally heterogeneous zones with stratified regions for which the reflection and transmission super-matrices are block diagonal. The relations (3.16) can be viewed as a wavenumber domain implementation of the results for reflection and transmission operators in heterogeneous media given by Kennett (1986).

We have seen that with suitable choices for the seismic field matrices it is possible to simplify the calculation of the reflection and transmission super-matrices. However, there remains the problem of constructing the extensions of the initially up and downgoing wavefields into the heterogeneous regions. In certain situations this can be done quite readily and then the propagation invariant route is of considerable computational merit.

### 3.2 Application to an irregular interface

A problem where the construction of the requisite field is reasonably straightforward is the situation of an irregular interface separating two uniform regions. The surfaces \( z_A \)
and $z_0$ can then be taken as the upper and lower bounds on the position of the interface. With an intermediate reference surface ($z = z_0$) the two field representations (3.1) and (3.2) can be continued to the reference level $z_0$ using plane wave expansions dependent on the shape of the interface, under the usual 'Rayleigh Ansatz' that the character of the field in the bounding half spaces is sustained on the interface itself. This assumption is a good approximation unless the variations of the interface are extreme in amplitude or slope (Aki & Richards 1980, section 13.4).

Since the field matrices $B_{DA}$, $B_{UA}$ and $B_{DB}$ all be continued to the reference level $z_0$, the reflection and transmission matrices $R_{DB}^D$, $T_{DB}^B$, $R_{UB}^U$ and $T_{UB}^B$ can be constructed with only a single large-scale matrix inversion.

We illustrate this procedure by application to the well-known test case introduced by Boore, Larner & Aki (1971). This consists of a sedimentary basin lying beneath a free surface, with a vertically incident $SH$-wavefield. The symmetrical basin varies from 1 km at the edge to 6 km in the centre with a shape defined by

$$h(x) = D + \frac{C}{2} \left[ 1 - \cos \left( \frac{2\pi}{w} \left( x - \frac{w}{2} \right) \right) \right], \quad (3.17)$$

where $D = 1$ km, $C = 5$ km and $w = 50$ km. Fig. 3 displays the $y$-component of surface displacement at the top of the basin calculated by three different methods with a common incident waveform of Ricker type

$$f(t) = \frac{\sqrt{\pi}}{2} \left( b^2 - \frac{1}{2} \right) \exp \left( -b^2 \right). \quad (3.18)$$

with $b = \pi(t - t_c)/p_c$, $t_c = 20$ s and $p_c = 18.3$ s.

The finite difference solution was calculated using the scheme described by Yamanaka, Seo & Samano (1989) in the time domain. The other two techniques are based on a discrete wavenumber representation and Fourier synthesis in frequency. By introducing propagator 'super-matrices', whose entries are the well-known single wavenumber propagators, Koketsu (1987) extended Aki & Larner's (1970) treatment for wavenumber interactions to multi-layered media with irregular boundaries, including the effect of the free surface. A similar formulation was independently discovered by Geli, Bard & Jullien (1988). The 'propagator' solution was calculated using Koketsu's method. The 'invariant embedding' approach makes use of the development in Section 3.1. Using a reference level at 1 km, the reflection and transmission super-matrices $R^1$, $T^1$ for $SH$-waves incident on the interface are constructed using (3.13), (3.14) and their analogues for upgoing waves. The surface response is then calculated by constructing the transmission effect to the surface including allowances for multiple reflections from the free surface:

$$T_{U}^n[I - R_{DU}^n]^{-1}T_{D}^n \quad (3.19)$$

where $T_{U}^n$ and $R_{U}^n$ are block diagonal and represent transmission through the uniform top 1 km and reflection from the free surface including appropriate phase delays.

As we see from Fig. 3, the results obtained using the three different techniques agree well and it is difficult to detect any difference between the propagator and invariant embedding results. At later times the differences in the finite difference solution arise from cumulative delays due to numerical dispersion. Koketsu (1987, fig. 2.5) has previously shown good agreement for the propagator method and a variety of other schemes for this laterally varying model, so that these calculations confirm the validity of the new approach based on propagation invariants.

On the Fujitsu VP-100 of the ANU Supercomputer Facility, the calculations for Fig. 3 took 25 s for the invariant embedding method, 30 s for the propagator technique and over 10 min for the finite difference code. The speed advantage of the invariant embedding approach derives from the simplification of the large-scale matrix algebra and is even more marked for multiple interfaces of $P$-$SV$-wave problems. With this new tool it should prove feasible to make a direct attack on 3-D problems.

In a companion paper (Koketsu, Kennett & Takenaka 1990) we will give detailed explanations of both the 'propagator' and 'invariant embedding' approaches with more complex examples.

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Woodfield invariants in laterally varying media