2-D reflectivity method and synthetic seismograms for irregularly layered structures—II. Invariant embedding approach

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SUMMARY
Laterally varying interfaces cause coupling between wavenumbers so that seismograms in two-dimensionally layered media can be synthesized by means of ‘supermatrices’, which include the coupled contributions of all the wavenumbers. We introduce reflection and transmission ‘supermatrices’ in order to eliminate numerical problems arising from loss of precision for evanescent waves in the seismogram synthesis. An interface is assumed to be such that the reflected and transmitted wavefields on its two sides can be represented as purely upgoing and downgoing waves, i.e. the Rayleigh ansatz is imposed. The computational demands of this method can be kept to a minimum by exploiting propagation invariants in the coupled wavenumber domain.

The superior performance of this ‘invariant embedding’ approach when compared to propagator or finite difference schemes is illustrated by application to the response of sedimentary basins to excitation by an incident plane wave or a line force. The results are in good general agreement with the other methods, but show greater numerical stability and computational efficiency. In the case of a single interface the ‘invariant embedding’ procedure for P–SV-waves takes 45 per cent less computation time and 29 per cent less memory than the propagator method of Koketsu (1987a,b). The gains are reduced in a multilayer case because of the level of computation required to calculate the addition rules for the large reflection and transmission supermatrices.

Key words: irregular layering, propagation invariant, synthetic seisograms.

1 INTRODUCTION
A wide range of theoretical studies have being carried out over the last 20 years with the object of synthesizing seismograms in irregularly layered media. Many authors have tried to match the boundary conditions of irregular interfaces by superposition of the known solutions in a homogeneous layer. If the solution \( u(x) \) in an irregularly layered medium is represented as a linear combination of the homogeneous solutions \( \phi_n(x) \),

\[
  u(x) = \sum_{n=1}^{N} a_n \phi_n(x),
\]

the matching of the boundary condition \( B(u) = 0 \) can be accomplished with ‘the method of weighted residuals’ (Finlayson 1972; Fletcher 1984). The weighting factors \( a_n \).

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with the collocation BIEM. Sánchez-Sesma, Herrera & Avilés (1982) used the least-squares BIEM for the same problem. Campillo & Bouchon (1985) computed scattered excitation by a line-source with the finite Fourier expansion ("discrete wavenumber representation") of the Green's function and collocation matching. The Boundary Element Method (BEM) looks quite different from BIEM, but it is also based on a similar concept (Breblick 1978), and Kawase (1988) computed scattered P–SV-waves using BEM.

However, computation based on the use of the Green's functions is very complicated and time consuming. If we can synthesize seismograms by using only homogeneous plane waves, the computation would become simple and fast. Aki & Lermer (1970) showed that this approach should be valid when we rely on the Rayleigh ansatz, i.e. an upcoming wavefield can be neglected in the lowermost half-space of an irregularly layered medium. Because of coupling between wavenumbers caused by an irregular interface they considered all the wavenumbers together. To handle this coupling Koketsu (1987a) introduced 'superpropagators' (Haines 1988), whose entries are the well-known single-wavenumber propagators, and extended Aki & Lermer’s formulation to multilayered media. A more complete development of this formulation appears in Koketsu (1987b), including P–SV interactions. Later, Geli, Bard & Jullien (1988) and Honke (1988) independently discovered similar formulations.

Axilrod & Ferguson (1990) reported that the CPU time required for the method of Campillo & Bouchon (1985) is approximately an order of magnitude greater than for this approach. The Boundary Element Method should also require a CPU time with the same order.

The propagator formulation for irregularly layered media is subjected to the class of numerical problems, which beset propagation techniques in horizontally layered media, especially loss of numerical precision for evanescent waves. Takenaka (1990) has shown how the reflection/transmission matrix approach of Kennett (1983) can be adapted to irregularly layered media. This procedure eliminates problems with evanescent waves, but requires a number of large-scale matrix inversions, which introduce a different range of numerical complications.

In this paper we will present an alternative formulation of the reflection/transmission matrix approach, which exploits the propagation invariants of Kennett, Koketsu & Haines (1990) to simplify the calculations. This new method will be called 'invariant embedding' based on the nomenclature in the review paper of Chin, Hedstrom & Thigpen (1984). We demonstrate the superior numerical stability and computational efficiency of the invariant embedding technique compared with the propagator formulation using a variety of numerical examples. In order to provide a clear link to the reflection and transmission results the notation used is based on Kennett (1983) rather than that of Koketsu (1987a, b).

2 IRREGULAR INTERFACES

We consider layered media with two-dimensionally varying interfaces. We take a Cartesian coordinate system \((x, y, z)\) with z-axis taken positive downward, and assume that the free surface and interfaces vary in the \(x-z\) plane. Therefore the wavefield and media do not depend on the coordinate \(y\).

We will assume a time dependence \(\exp (i \omega t)\), but will not normally represent the time variation explicitly.

In a homogeneous and isotropic layer having P-wave velocity \(\alpha\), S-wave velocity \(\beta\) and density \(\rho\), the displacement can be expressed as

\[
\mathbf{u}(x, z) = [u(x, z), w(x, z), v(x, z)]^T
\]

\[
= \left( \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial x}, 0 \right)^T + \left( -\frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial x}, 0 \right)^T + (0, 0, v)^T, \tag{2.1}
\]

in terms of \(P, S\) and \(SH\) contributions (note that we locate \(v\) at the last element to isolate the \(SH\) contribution). When we take Fourier transforms with respect to \(x\)

\[
\mathbf{f}(k, z) = \int_{-\infty}^{\infty} f(x, z) e^{-ikx} \, dx,
\]

(2.1) has harmonic solutions in the \(k\) (horizontal wavenumber) domain

\[
\tilde{\phi}(k, z) = \exp (\pm iv_k z),
\]

\[
\tilde{\psi}(k, z) = \exp (\pm iv_k z),
\]

\[
\tilde{v}(k, z) = \exp (\pm iv_k z),
\]

where

\[
\nu_k = (k_x^2 - k_z^2)^{1/2} \quad (k_z = \omega/c, c = \alpha or \beta)
\]

is a vertical wavenumber. We can then write general solutions with weighting factors \(P_{U,D}, S_{U,D}\) and \(H_{U,D}\) as

\[
\tilde{\phi}(k, z) = P_{U,D} \exp (\pm iv_k z) + P_{D} \exp (\pm iv_k z),
\]

\[
\tilde{\psi}(k, z) = S_{U,D} \exp (\pm iv_k z) + S_{D} \exp (\pm iv_k z),
\]

(2.5)

(2.3)

where

\[
\phi(k, z) = \tilde{\phi}(k, z) \exp (iv_k z),
\]

\[
\psi(k, z) = \tilde{\psi}(k, z) \exp (iv_k z),
\]

\[
v(k, z) = \tilde{v}(k, z) \exp (iv_k z),
\]

(2.4)

We have a free choice of the scaling parameters \(e_p, e_s\) and \(e_H\) in (2.5), because they affect only the physical meanings of the quantities \(P_{U,D}, S_{U,D}\) and \(H_{U,D}\) (Kennett 1983, chapter 3).

We define the upgoing and downgoing wave vectors as

\[
u_{U,D} = [P_{U,D} \exp (\pm iv_k z), S_{U,D} \exp (\pm iv_k z), H_{U,D} \exp (\pm iv_k z)]^T
\]

(2.6)

and insert (2.5) into the Fourier transform of (2.1). Then we have

\[
\tilde{\mathbf{u}}(k, z) = [\tilde{\mathbf{u}}(k, z), \tilde{\mathbf{u}}(k, z), \tilde{\mathbf{u}}(k, z)]^T = \mathbf{M}^0_{U,D} \nu_{U} + \mathbf{M}^0_{D,D} \nu_{D},
\]

(2.7)

where

\[
\mathbf{M}^0_{U,D}(k) = \begin{pmatrix}
ike_p & +iv_k e_s & 0 \\
 +iv_k e_p & ike_s & 0 \\
0 & 0 & ike_H
\end{pmatrix}.
\]

(2.8)

Similarly, by using the stress–strain relation we find that

\[
\tilde{\mathbf{e}}_{x}(k, z) = [\tilde{\mathbf{e}}_{xx}(k, z), \tilde{\mathbf{e}}_{xz}(k, z), \tilde{\mathbf{e}}_{yx}(k, z)]^T
\]

\[
= N^x_{U,D} \nu_{U} + N^x_{D,D} \nu_{D},
\]

(2.9)

where

\[
N^x_{U,D}(k) = \begin{pmatrix}
\mu e_p & \pm 2\mu k v e_s & 0 \\
\mp 2\mu k v e_p & -\mu e_s & 0 \\
0 & 0 & ike_H
\end{pmatrix}.
\]

(2.10)
\[ N_{zU,D}(k) = \begin{pmatrix} \pm 2\mu k\nu \varepsilon_p & -\mu k \varepsilon_S & 0 \\ \mu k \varepsilon_p & \pm 2\mu k\nu \varepsilon_S & 0 \\ 0 & 0 & \pm \mu \nu \varepsilon_H \end{pmatrix}, \]

with

\[ r = 2\nu_a^2 - k_p^2, \quad l = 2k^2 - k_p^2 \]

The form of the matrices in (2.8) and (2.10) shows the decoupling between the \(P-SV\)-wavefield \( (P_{U,D},S_{U,D}) \) and the \(SH\)-wavefield \( (H_{U,D}) \) in homogeneous layers. We note that

\[ [M_{10}^0(-k)]^T N_{zU,D}(k) - [N_{zU,D}(k)]^T M_{10}^0(k) = \theta \]

and

\[ [M_{10}^0(-k)]^T N_{dU}(k) = [N_{zU,D}(k)]^T M_{10}^0(k) \]

\[ = -[M_{10}^0(-k)]^T N_{dU}(k) + [N_{zU,D}(k)]^T M_{10}^0(k) \]

\[ = \text{diag}(2\mu k^2\nu \varepsilon_p^2, 2\mu k^2\nu \varepsilon_S^2, 2\mu \nu \varepsilon_H^2). \]

To simplify the subsequent development we take

\[ c_p = (2\mu k^2\nu \varepsilon_p)^{-1/2}, \quad c_S = (2\mu k^2\nu \varepsilon_S)^{-1/2}, \quad c_H = (2\mu \nu \varepsilon_H)^{-1/2}, \]

so that (2.13) is further reduced to an imaginary identity matrix.

Consider now layers \(A\) and \(B\) separated by an irregular interface with the shape controlled by

\[ z(x) = z_0 + h(x), \]

which fluctuates around the reference level \(z_0\) (Fig. 1). The boundary condition of welded contact at the interface \(z(x)\) requires the continuity of the displacement \( u[x, z(x)] \) and the traction \( t[x, z(x)] \) across the interface. In terms of the wavenumber components we need

\[ \int \{\tilde{u}[k, z(x)], \tilde{v}[k, z(x)]\}^T e^{i k x} dk \]

(2.16)

to be continuous to satisfy these two interface conditions. We assume that the wavefield on the two sides of the irregular interface can be represented by the homogeneous solutions (2.3). Then, inserting (2.7) into the displacement part of (2.16) and redefining the wave vectors to leave only \(x\)-independent terms, i.e.

\[ u_{U,D}(k) = [P_{U,D} \exp (\pm iv_0 z_0), S_{U,D} \exp (\pm iv_0 z_0), \]

\[ H_{U,D} \exp (\pm iv_0 z_0)]^T, \]

we obtain

\[ \int \{\tilde{u}[k, z(x)], \tilde{v}[k, z(x)]\}^T e^{i k x} dk = \int [M_{U}(k) v_U(k) + M_O(k) u_O(k)] dk \]

(2.18)

where

\[ M_{U,O}(k) = M_{10}^0(k) E_{U,O}[h(x)] e^{i k x}, \]

\[ E_{U,O}(z) = \text{diag} \{ \exp (\pm iv_0 z), \exp (\pm iv_0 z), \exp (\pm iv_0 z) \}. \]

On the other hand, by taking \( n = [n_x(x), n_z(x)]^T \) as the unit normal to the interface, the traction can be derived from the stress as

\[ t(x, z) = n_x(x) \sigma_x(x, z) + n_z(x) \sigma_z(x, z). \]

(2.20)

\[ n = [n_x, n_y] \]

Figure 1. An irregular interface between layers \(A\) and \(B\). \( n \) is the unit normal to the interface.

Inserting this expression and (2.9) into the traction part of (2.16) we get

\[ \int [N_{U}(k) v_U(k) + N_O(k) v_O(k)] dk, \]

(2.21)

where

\[ N_{U,O}(k) = [n_x(x) N_{0U,D}^0(k) + n_z(x) N_{0U,D}^0(k)] E_{U,O}[h(x)] e^{i k x}. \]

We note that

\[ n_x(x) = -h'(1 + h'^2)^{1/2}, \quad n_z(x) = 1/(1 + h'^2)^{1/2}, \]

(2.23)

where \( h' = dh(x)/dx \) and Kennett (1972) found that \((1 + h'^2)^{1/2}\) is common to the traction forms in both layers \(A\) and \(B\). Thus we can omit this factor from \(N_{U,O}(k)\) and set

\[ N_{U,D}(k, x) = [-h'(1 + h'^2)^{1/2}, n_z(x) N_{0U,D}^0(k)] E_{U,D}[h(x)] e^{i k x}. \]

(2.24)

\[ M_{U,D} \] in (2.19) and \( N_{U,D} \) in (2.24) keep the same matrix form as \( M_{10}^0, N_{0U,D} \) and \( N_{0U,D}^0 \) so that the decoupling between \(P-SV\)- and \(SH\)-waves is still valid even in two-dimensionally layered media.

The boundary condition of continuity of displacement and traction can now be written as

\[ \int D_x(x, k) v_x(x) dk = \int D_y(x, k) v_y(x) dk, \]

(2.25)

where \(D(x, k)\) and \(v(k)\) are a pseudo-eigenvector matrix and a total vector defined as

\[ D(x, k) = \begin{pmatrix} M_{U}(x, k) & M_{O}(x, k) \\ N_{U}(x, k) & N_{O}(x, k) \end{pmatrix}, \quad v(k) = \begin{pmatrix} v_U(k) \\ v_O(k) \end{pmatrix}. \]

(2.26)

If the interface is flat, i.e. \( h(x) = 0 \), the integrands in (2.25) are independent of \(x\) except for \(e^{i k x}\), and the integral equation (2.25) will simply be solved as

\[ \begin{pmatrix} M_{U,A}(k) & M_{O,A}(k) \\ N_{U,A}(k) & N_{O,A}(k) \end{pmatrix} v_A(k) = \begin{pmatrix} M_{U,B}(k) & M_{O,B}(k) \\ N_{U,B}(k) & N_{O,B}(k) \end{pmatrix} v_B(k). \]

(2.27)

For an irregular interface, however, we have no trivial solutions like (2.27). In other words, scattering by the irregular interface causes the coupling between different wavenumbers so that we have to consider all the wavenumbers together.

In order to solve the integral equation (2.25) we first
replace the infinite integrals with finite sums ('a discrete wavenumber representation') and set
\[
\int \mathbf{D}(x, k)u(k) \, dk = \sum_i \mathbf{D}(x, k_i)u(k_i)
\]
(2.28)
where \( \mathbf{D} \) includes \( dk \) on the right-hand side. We next introduce the total wave 'supervector' \( \mathbf{v} \) and the continuous 'supermatrix' \( \mathbf{D}(x) \) whose entries for wavenumber \( k_i \) are just \( u(k_i) \) and \( \mathbf{D}(x, k_i) \). We can then express (2.25) as
\[
\mathbf{D}_A(x)v_A = \mathbf{D}_B(x)v_B.
\]
(2.29)
(2.29) represents the boundary condition on the irregular interface (2.15) as a set of continuous equations in terms of \( x \), but can be discretized by using a weighting scheme as in (1.3). For example, using collocation matching we get
\[
\mathbf{D}_A(x)v_A = \mathbf{D}_B(x)v_B.
\]
(2.30)
However, Aki & Larner (1970) adopted a slightly different approach: they took the Fourier transform (2.2) of the both sides of (2.29) with respect to a set of wavenumbers and so derived
\[
\mathbf{D}_A(k_j)v_A = \mathbf{D}_B(k_j)v_B,
\]
(2.31)
By introducing the discrete supermatrix \( \mathbf{D} \) in the coupled wavenumber domain, whose entry for \( k_j \) is just \( u(k_j) \) and \( \mathbf{D}(k_j) \), we rewrite (2.31) as
\[
\mathbf{Dv}_A = \mathbf{Dv}_B.
\]
(2.32)
\( \mathbf{D} \) represents interconversion between wavenumbers \( k_j \) and \( k_i \)'s. If the interface is horizontal as \( h(x) = 0 \), its elements vanish except for \( k_j = k_i \) so that \( \mathbf{D} \) becomes a block-diagonal matrix, whose blocks have only diagonal parts consisting of the elements of \( \mathcal{M}_{ul}^0 \) and \( \mathcal{N}_{ul}^0 \). Even for an irregular, but smooth interface, significant elements of \( \mathbf{D} \) are concentrated in bands near the block diagonals. The widths of the bands are roughly proportional to the irregularity of the interface. As a result the supermatrix \( \mathbf{D} \) constructed from \( \mathbf{D} \) is more stable in numerical computations than the alternative system derived from the usual collocation matching in the space domain. The approach of Aki & Larner (1970) can thus be called 'spectral collocation matching'.

We now introduce the displacement-traction vector
\[
\mathbf{b}[x, z(x)] = \{u[x, z(x)], (1 + h'^2)^{1/2}[x, z(x)]\}^T
\]
(2.33)
for the interface (2.15), and recall the equations (2.15)-(2.26) so that
\[
\mathbf{b}[x, z(x)] = \{u[x, z(x)], (1 + h'^2)^{1/2}[k, z(x)]\}^T e^{i\mathbf{k} \cdot \mathbf{z}} \, dk
\]
(2.34)
By taking the Fourier transform of both sides of (2.34), we can write
\[
\mathbf{b}(k_j, z_0) = \hat{\mathbf{D}}(k_j)\mathbf{v}_{z_0}, \quad j = 1, 2, \ldots
\]
(2.35)
or
\[
\mathbf{b}(z_0) = \mathbf{Dv}(z_0),
\]
(2.36)
where the supervector \( \mathbf{b}(z_0) \) is constructed from \( \mathbf{b}(k_j, z_0) \). Thus the continuity relation (2.32) can be further simplified to
\[
\mathbf{b}_A(z_0) = \mathbf{b}_B(z_0).
\]
(2.37)
Consider finally that a homogeneous layer is bounded by two irregular interfaces at the levels \( z_1 \) and \( z_2 \) (Fig. 3). From (2.36) \( \mathbf{b}(z_1) = \mathbf{D}_{1+} \mathbf{v}(z_1) \) immediately beneath \( z_1 \), and \( \mathbf{b}(z_2) = \mathbf{D}_{2-} \mathbf{v}(z_2) \) immediately above \( z_2 \). Since \( \mathbf{v}(z_2) \) can be connected to \( \mathbf{v}(z_1) \) with the phase supermatrix \( \mathbf{E}(z) \), constructed from \( \mathcal{E}_{ul}(z) \) in (2.19), as
\[
\mathbf{v}(z_2) = \mathbf{E}(z_2 - z_1)\mathbf{v}(z_1),
\]
(2.38)
we obtain
\[
\mathbf{b}(z_2) = \mathbf{P}_{21} \mathbf{b}(z_1), \quad \mathbf{P}_{21} = \mathbf{D}_{2-} \mathbf{E}(z_2 - z_1) \mathbf{D}_{1+}^{-1}
\]
(2.39)
\( \mathbf{P}_{21} \) is a propagator supermatrix of a 'super propagator' (Haines 1988) for an irregularly layered medium. Koketsu (1987a,b) computed synthetic seismograms in a two-dimensionally layered medium by using this super propagator. However, the form (2.39) includes the phase supermatrix holding the terms \( \exp(\pm iv \cdot z) \), which may grow exponentially when the wavefield is evanescent. This can

Figure 2. Configuration of a matrix, continuous supermatrix and discrete supermatrix.

Figure 3. A homogeneous layer lying between irregular interfaces at depths \( z_1 \) and \( z_2 \).
lead to numerical instability in case of high frequencies, low phase velocities or thick layers. In addition it is necessary to compute a full inverse of $D$. $D$ is $8N_i \times 8N_i$ for P-SV-waves, or $4N_i \times 4N_i$ for SH-waves ($N_{ij}$ is the number of wavenumbers $k_{ij}$). Such a large-scale matrix inversion may cause computational inaccuracy even if $D$ is nearly block-diagonal.

3 REFLECTION, TRANSMISSION AND INVARIANTS

For horizontally layered media Kennett (1983) has shown that the exponentially growing terms can be avoided if we formulate the wavefield in terms of the reflection and transmission matrices rather than the propagators. Take-naka (1990) has recently extended this approach to irregularly layered media by direct analogy with the flat interface results of Kennett (1983), but with substantially higher dimensionality.

Consider an incident downgoing wave from the layer $A$ into the layer $B$ through an irregular interface (Fig. 4a). This will give a reflected upgoing wave in $A$ and transmitted downgoing wave in $B$. If we rely on the Rayleigh ansatz (see e.g. Aki & Richards 1980, chapter 13), there will be no upgoing wave in $B$. We partition the pseudo-eigenvector supermatrix $D$ and the total wave supervector $v$ as in (2.26):

$$D = \begin{pmatrix} M_U & M_D \\ N_U & N_D \end{pmatrix}, \quad v = \begin{pmatrix} v_U \\ v_D \end{pmatrix}.$$  

and then rewrite boundary condition (2.32) as

$$\begin{pmatrix} M_U & M_D \\ N_U & N_D \end{pmatrix} \begin{pmatrix} v_{UA} \\ v_{DA} \end{pmatrix} = \begin{pmatrix} M_{UB} & M_{DB} \\ N_{UB} & N_{DB} \end{pmatrix} \begin{pmatrix} v_{UB} \\ v_{DB} \end{pmatrix}.$$  

(3.2)

By defining the reflection and transmission supermatrices $R_D, T_D$ to connect the partitioned wavefields:

$$v_{UA} = R_D v_{DA}, \quad v_{DB} = T_D v_{DA}.$$  

(3.3)

Figure 4. Schematic representation of reflected and transmitted waves due to (a) downgoing or (b) upgoing wave incidence on the irregular interface separating layers $A$ and $B$.

we obtain

$$\begin{pmatrix} M_{UA} & M_{DA} \\ N_{UA} & N_{DA} \end{pmatrix} \begin{pmatrix} R_D \\ I \end{pmatrix} = \begin{pmatrix} M_{UB} & M_{DB} \\ N_{UB} & N_{DB} \end{pmatrix} \begin{pmatrix} T_D \\ 0 \end{pmatrix},$$  

(3.4)

where $I$ and $0$ are the identity and zero supermatrices. Similarly, if we consider an incident upgoing wave from $B$ into $A$ (Fig. 4b) and define the supermatrices $R_U, T_U$ as

$v_{DB} = R_U v_{UB}, \quad v_{UA} = T_U v_{UB},$  

(3.5)

then (2.32) becomes

$$\begin{pmatrix} M_{UA} & M_{DA} \\ N_{UA} & N_{DA} \end{pmatrix} \begin{pmatrix} T_U \\ I \end{pmatrix} = \begin{pmatrix} M_{UB} & M_{DB} \\ N_{UB} & N_{DB} \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}$$  

(3.6)

by neglecting the downgoing wavefield in $A$.

Take-naka (1990) solved the simultaneous linear equations in (3.4) and (3.6) explicitly, and found that

$$T_D = M_{Db}^{-1} S_{dd} M_{Da}, \quad R_D = M_{Ub}^{-1} S_{ud} M_{Da},$$  

(3.7)

$$R_U = M_{Db}^{-1} S_{dd} M_{Ub}, \quad T_U = M_{Ub}^{-1} S_{ud} M_{Ub},$$  

where

$$S_{dd} = -(A_{db} - A_{ua})^{-1}(A_{ua} - A_{da}),$$  

$$S_{ud} = +(A_{ua} - A_{db})^{-1}(A_{db} - A_{da}),$$  

$$S_{ud} = +(A_{da} - A_{db})^{-1}(A_{db} - A_{da}),$$  

(3.8)

$$S_{uu} = -(A_{da} - A_{db})^{-1}(A_{db} - A_{da}),$$  

$$A_{ud} = N_{ud}(M_{ud})^{-1}.$$  

The calculation of $R_{ud}, T_{ud}$ by the above formulation requires five matrix inversions, i.e. $M_{ua}^{-1}, M_{da}^{-1}, M_{ub}^{-1}, M_{db}^{-1}$ and $(A_{da} - A_{db})^{-1}$. Take-naka (1990) also found the explicit expressions for the continuous supermatrices $A_{ud}(x, k)$, so then was able to reduce the number of matrix inversions, but, even in this case, three inversions still remain.

Kennett et al. (1990) established a spatial propagation invariant

$$\int \left[ u_1^T(x, z) t_2(x, z) - t_1^T(x, z) u_2(x, z) \right] dS$$  

(3.9)

for two elastic displacement fields $u_1$ and $u_2$ and their associated traction fields $t_1$ and $t_2$. This will be invariant for any surface spanning the $x$-$y$ plane in a laterally heterogeneous and anisotropic medium. If we take the interface $z(x) = z_0 + h(x)$ in a two-dimensionally layered medium as $S$, $dS = (1 + h^2) \frac{dx}{2}$, so that (3.9) yields

$$\int b_1^T[x, z(x)] N b_2[x, z(x)] dx, \quad N = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$  

(10.10)

If we assume the interchangeability of integrals, we can note that

$$\int f(x) g(x) dx = \frac{1}{(2\pi)^2} \int \int dk' dk \tilde{f}(k') \tilde{g}(k) dk' dx e^{ik'x}$$  

$$= \frac{1}{2\pi} \int \int dk' dk \tilde{f}(k') \tilde{g}(k) \delta(k' + k)$$  

$$= \frac{1}{2\pi} \int (-k) \tilde{g}(k) dk.$$  

(11.11)
By using (3.11) and replacing the infinite integral with a finite sum ('a discrete wavenumber representation') we rewrite the invariant (3.10) as

$$\tilde{b}^T(-(k, z_0)N\Delta(k, z_0) dk = b_i^T(z_0)Nb_2(z_0), \quad (3.12)$$

where N is a supermatrix made from N.

The operator $\#$, which was first introduced by Haines (1988), performs transposition of a supervector or supermatrix and switches the sign of the wavenumbers for their elements. All the supermatrices appearing in this paper have a partitioned form such as

$$G = \begin{pmatrix} G_{PP} & G_{PS} & 0 \\ G_{SP} & G_{SS} & 0 \\ 0 & 0 & G_{HH} \end{pmatrix}. \quad (3.13)$$

$G^\#$ can be also partitioned after transposition with the operator $\#$ as

$$G^\# = \begin{pmatrix} G_{PP}^\# & G_{PS}^\# & 0 \\ G_{SP}^\# & G_{SS}^\# & 0 \\ 0 & 0 & G_{HH}^\# \end{pmatrix}. \quad (3.14)$$

When we define the Fourier transform as in (2.2) and discretize it with an equal interval $\Delta k$, supervectors and the partitions of supermatrices have the forms illustrated in Fig. 5a. Since they include $G$ about the line $k = 0$. $G^\#$ is that of $G$ about the line $k_i + k_0 = 0$ (Fig. 5b), which is the alternate diagonal of $G$ to the line $k = k_0$ (the reflection line of $G^\dagger$).

Therefore, the operator $\#$ has similar characteristics to the transpose operator $T$:

$$(G^\dagger)^\# = G, \quad (GH)^\# = H^\#G^\#, \quad (G^{-1})^\# = (G^\#)^{-1}. \quad (3.15)$$

These relations are valid not only for the partitions of supermatrices but also for supermatrices themselves.

In the previous section we have already established the upgoing and downgoing wavefields as $u_1$ and $u_2$ in an irregularly layered medium. Thus, from (2.36) and (3.1) we find that the supervectors of up and downgoing waves

$$b^+ = (M^u, N^u)^T v_U = B^u v_U, \quad b^D = (M^D, N^D)^T v_D = B^D v_D, \quad (3.16)$$

work as $b_1$ and $b_2$ in (3.12). We now construct some useful identities for solving (3.4) and (3.6) by means of the invariant (3.12) and the wavefields (3.16). Consider two surfaces $S_1$ and $S_2$ spanning the $x-y$ plane close to an irregular interface at the level $z_0$. $S_1$ is a horizontal plane immediately above $z_0$, while $S_2$ is located on the $S_1$ side along the interface (Fig. 6). If we take $b_1 = b_2 = b_U$, then

$$b_U^*(z_0)Nb_2(z_0) = v_U^\#(b_U, b_U)^T v_U, \quad (3.17)$$

where

$$(B_1, B_2) = B_1^\#NB_2 = M_1^uN_2 - N_1^uM_2. \quad (3.18)$$

Since $S_1$ is horizontal, $M^u$ and $N^u$ on $S_1$ are block-diagonal supermatrices, whose entries are the elements of $M_{UU}^u$ and $N_{UU}^u$. By recalling (2.12), i.e.

$$[M_{UU}^u(-k)]^N x_{U, D}(k) - [N_{UU}^u(-k)]^TM_{UU}^u(k) = \theta, \quad (3.19)$$

we find that every entry of $(B_1, B_2)$ vanishes on $S_1$. This argument is also valid in the case of $b_1 = b_2 = b_D$, so that on $S_1$

$$(B_U, B_U) = (B_D, B_D) = 0. \quad (3.19)$$

In a similar way from (2.13) and (2.14) we obtain another identity on $S_1$:

$$(B_D, B_U) = -(B_U, B_D) = i\mathbf{1}. \quad (3.20)$$

Meanwhile, $b_U^\#Nb_2$ is invariant at any surface spanning the $x-y$ plane, so that

$$v_{U, D}((B_U, B_U)) = -(B_U, B_D)S_1 v_{U, D} = 0, \quad (3.21)$$

where

$$(B_D, B_U)S_1 = -(B_D, B_U)S_1 v_{U, D} = 0. \quad (3.21)$$

Since (3.21) has to be satisfied for arbitrary supervectors $v_{U, D}$, the identities (3.19) and (3.20) should be valid even on $S_2$ along the irregular interface.

These identities mean that suitable choice of $B$ in forming an invariant with the two sides of (3.4) or (3.6) can readily
extract either $R_{U,D}$ or $T_{U,D}$. Firstly, (3.4) multiplied by

$$[B_{UA}, -B_{DB}]^\mu = \begin{pmatrix} -\text{N}_{UB} & M_{UB} \\ -\text{N}_{UA} & M_{UA} \end{pmatrix}$$

(3.22)
leads to

$$\begin{pmatrix} \langle B_{DB}, B_{UA} \rangle & \langle B_{DB}, B_{DA} \rangle \\ 0 & -i \end{pmatrix} \langle R_D \rangle = \begin{pmatrix} i \langle B_{UA}, B_{UB} \rangle & \langle B_{UA}, B_{DB} \rangle \\ 0 & 1 \end{pmatrix} \langle T_D \rangle.$$

(3.23)
and hence

$$R_D = -\langle B_{DB}, B_{UA} \rangle^{-1} \langle B_{DB}, B_{DA} \rangle,$$

(3.24)
and

$$T_D = -i \langle B_{UA}, B_{DB} \rangle^{-1}.$$

Secondly, (3.6) multiplied by $[B_{DB}, -B_{UA}]^\mu$ yields

$$R_U = -i \langle B_{DB}, B_{UA} \rangle^{-1} \langle B_{DB}, B_{UB} \rangle,$$

(3.25)
and

$$T_U = +i \langle B_{DB}, B_{UA} \rangle^{-1}.$$

However, from (3.15) we know that

$$\langle B_{UA}, B_{UB} \rangle^{-1} = -\langle (B_{DB}, B_{UA})^{-1} \rangle^\mu,$$

(3.26)
and so only a single matrix, i.e. $\langle B_{DB}, B_{UA} \rangle$ need be inverted to generate $R_{U,D}$ and $T_{U,D}$. Besides $\langle B_{DB}, B_{UA} \rangle$ is just $4N_x \times 4N_x$ for $P-$SV-waves or $2N_x \times 2N_x$ for $SH$-waves, while a super propagator $D$ is $4N_x \times 4N_x$.

The above calculation of $R_{U,D}$ and $T_{U,D}$ may sometimes be numerically unstable for the normalization (2.14) due to the influence of the complex roots. Therefore we first calculate $R_{U,D}$ and $T_{U,D}$ without this normalization, and then obtain $R_{U,D}$ and $T_{U,D}$ by

$$R_D = \varepsilon_A^\mu R_B \varepsilon_A, \quad T_D = \varepsilon_B^\mu T_U \varepsilon_A^{-1},$$

$$R_U = \varepsilon_B^\mu R_U \varepsilon_B, \quad T_U = \varepsilon_A^\mu T_U \varepsilon_B^{-1},$$

(3.27)
where $\varepsilon_A, B$ is a block-diagonal supermatrix consisting of $\varepsilon_p, \varepsilon_s$ and $\varepsilon_I$ in (2.14) for the layer $A$ or $B$.

When we compare (3.24) and (3.25) with (5.19) and (5.24) in Kennett (1983), we find that $T_{U,D}$ has an opposite sign to the corresponding transmission 'matrix' for a horizontally layered medium. This is due to a subtle difference in the definitions of $M_{U,D}$ and $N_{U,D}$. We defined them in (2.7) and (2.9) as matrices connecting potential amplitudes to displacements and stresses, while Kennett (1983) regarded them as eigenvector matrices of the equation of motion. However, by following the procedure in chapter 5 of Kennett (1983) we can prove the symmetries

$$R_D = R_D^\mu, \quad R_U = R_U^\mu, \quad T_U = T_D^\mu,$$

(3.28)
which are very similar to (5.60) in his book.

The direct analogy with the results for horizontally layered media (Kennett 1983, chapter 6) means that we can find the reflection and transmission supermatrices for a region $(z_A, z_C)$ in terms of the supermatrices for the subregions $(z_A, z_B)$ and $(z_B, z_C)$:

$$R_{BC}^C = R_{AB}^D + R_{AB}^D R_{BC}^D [1 - R_{AB}^D R_{BC}^D]^{-1} T_{AB}^D,$$

$$T_{BC}^C = T_{AB}^D [1 - R_{AB}^D R_{BC}^D]^{-1} T_{AB}^D,$$

$$R_{U}^C = R_{U}^D + R_{AB}^D R_{AB}^D [1 - R_{AB}^D R_{AB}^D]^{-1} T_{U}^C,$$

$$T_{U}^C = T_{U}^D [1 - R_{AB}^D R_{AB}^D]^{-1} T_{U}^C.$$

(3.29)

These addition rules include two supermatrix inversions, i.e. $(I - R_{AB}^D R_{BC}^D)^{-1}$ and $(I - R_{U}^D R_{U}^D)^{-1}$, but one of them can be calculated from the other by using

$$(I - R_{D}^D R_{U}^D) = (I - R_{U}^D R_{BC}^D) \mu.$$

(3.30)

For example we consider the case of the homogeneous layer lying between irregular interfaces illustrated in Fig. 3 as a single element of an irregularly layered medium. We suppose the reflection and transmission supermatrices at $z_2$ are known and write e.g. $R_D(z_2-)$, $T_D(z_2-)$, and $R_U(z_2-)$, $T_U(z_2-).$ Since the reflection and transmission supermatrices in the homogeneous layer are simply given by

$$R_D = R_U = 0, \quad T_D = T_U = E_D^{12},$$

(3.31)
where $E_D^{12}$ is a downgoing part of the phase supermatrix $E(z_2 - z_1)$, we can express the reflection and transmission supermatrices at $z_1+$ as

$$R_U(z_1+) = E_D^{12} R_D(z_2-) E_D^{12}, \quad T_U(z_1+) = T_D(z_2-) E_D^{12},$$

$$R_U(z_1+) = E_D^{12} R_D(z_2-) E_D^{12}, \quad T_U(z_1+) = T_D(z_2-) E_D^{12}.$$

(3.32)

The upgoing phase term exp $(+iwz)$ never appears in (3.32) so that the computation for $R_{U,D}$ and $T_{U,D}$ does not suffer exponential overflows.

4 SEISMOGRAM SYNTHESIS

We may continue to exploit the analogies with the case of horizontally layered media to generate convenient forms for the displacement field which can be used to synthesize seismograms. Consider the two-dimensionally layered medium illustrated in Fig. 7 with a line source at depth $z_S$ and a uniform half-space beneath the lowermost interface at level $z_L$. The surface displacement in the coupled wavenumber domain at level $z_0$ can be calculated using the analogue of (7.36) in Kennett (1983). In terms of the reflection and transmission supermatrices for the regions above and below the source the displacement

$$u_0 = \tilde{w}(I - R_{DS}^{SS})^{-1} T_{DS}^{SS} (I - R_{DS}^{SS} R_{DS}^{SS})^{-1} [\Sigma_D + R_{DS}^{SS} \Sigma_D],$$

(4.1)
where

$$R_{DS}^{SS} = R_{DS}^{SS} + T_{DS}^{SS} R_{DS}^{SS}(I - R_{DS}^{SS} R_{DS}^{SS})^{-1} T_{DS}^{SS},$$

(4.2)
Figure 7. Configuration of elastic half-space with a source at depth $z_S$ and receivers on the free surface at level $z_0$. Beneath the lowermost interface at level $z_L$, the medium has uniform properties.
and $\Sigma_{U,D}$ represents the outgoing or downgoing part of the jump supervector due to the source. The free-surface reflection supermatrix $\mathbf{R}$ and the surface amplification supermatrix $\mathbf{W}$ are defined by

$$\mathbf{R} = -\mathbf{N}^{-1}N_{z0}, \quad \mathbf{W} = M_{z0} + M_{Nz0}\mathbf{R}. \quad (4.3)$$

These quantities are best calculated in unnormalized form, and the normalization supermatrix $\mathbf{E}_0$ can be applied to give

$$\mathbf{R} = \mathbf{E}_0^{-1} \mathbf{R}\mathbf{E}_0, \quad \mathbf{W} = \mathbf{W}\mathbf{E}_0. \quad (4.4)$$

The recursive application of the addition rules (3.29) allows us to calculate $\mathbf{R}_{U,D}$ and $\mathbf{T}_{U,D}$ in the subregions $(z_0, z_0')$ and $(z_0, z_L)$, where an arbitrary number of layers may exist.

By means of (4.1) we can construct the response to a two-dimensionally layered medium in the transform domain, as a function of frequency $\omega$ and wavenumber $k$. To get synthetic seismograms we numerically invert the transforms.

We now show a variety of numerical examples to demonstrate the superior performance of our new invariant embedding approach compared with the propagator technique (Koketsu 1987a,b). First we calculate the seismic response of a layered sedimentary basin to an incident plane wave. In this case $z_L = z_1$, and $\Sigma_{D} = 0$. We can then calculate the surface displacement using

$$\mathbf{u}_0 = \mathbf{W}[1 - \mathbf{R}_{10}^{-1}\mathbf{R}^{-1}]\mathbf{T}_{10}\Sigma_{U}. \quad (4.5)$$

The first example is the simplest one where a uniform basin ($\alpha = 1.4 \text{ km s}^{-1}, \beta = 0.7 \text{ km s}^{-1}, \rho = 2.0 \text{ g cm}^{-3}$) lies above a half-space ($\alpha = 6.06 \text{ km s}^{-1}, \beta = 3.5 \text{ km s}^{-1}, \rho = 3.3 \text{ g cm}^{-3}$) with a vertically incident $SH$-wave. This is a variation of the well-known test case introduced by Boore, Larner & Aki (1971), but the interface has an asymmetrical shape with sharp corners. The depth of the basin $h(x)$ varies from 1 km at the edge to 6 km near the centre:

$$h(x) = D + C/2 \left\{ 1 - \cos \left[ 2\pi \left( x - \frac{w}{2} \right)/w \right] \right\}. \quad (4.8)$$

where $D = 0\text{ km}$, $C = 1\text{ km}$ and $w = 10\text{ km}$. This is a test

Figure 8. The $y$-component response of a uniform sedimentary basin to a vertically incident $SH$-wave on the free surface. The basin shown on the upper right side has an asymmetric shape and sharp corners.
Table 1. Computation parameters.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Wavefield</th>
<th>Source</th>
<th>Interfaces</th>
<th>( N, \Delta x(km) )</th>
<th>( N, \Delta t(s) )</th>
<th>( f_{\text{max}}(\text{Hz}) )</th>
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<tbody>
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<td>plane wave 1</td>
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<td>plane wave 3</td>
<td>2(^7) 5 2(^8)</td>
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<tr>
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<td>SH</td>
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<td>2(^7) 0.75 2(^8)</td>
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<td>1.5</td>
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Finite difference

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<th>Grid size(km)</th>
<th>Time steps</th>
<th>Step size(s)</th>
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\( N, \Delta x \): number of sampling or sampling interval of the space-domain FFT

\( N, \Delta t \): number of sampling or sampling interval of the time-domain FFT

\( f_{\text{max}} \): maximum frequency to be taken into calculation

Case introduced by Bard & Bouchon (1980). Fig. 9 displays the vertical component seismograms on the free surface calculated by IE and PR with a common incident Ricker wavelet of \( t_p = 5 \) s and \( t_r = 2.8 \) s. The PR calculation was carried out using the method of Koketsu (1987b). For this example IE is 45 per cent faster and takes 29 per cent smaller memory than PR.

In Fig. 9 we can again observe the lateral propagation of surface waves, but they are Rayleigh waves rather than Love waves in Fig. 8. When we look at the seismograms carefully, small differences can be found, especially in the later portions of the seismograms at distances of 0 km and 2.4 km. In the construction of the IE and PR seismograms truncation error is introduced in slightly different ways and this leads to the discrepancies in the forms of the seismograms. In Section 2 we have replaced the infinite integrals with finite sums in order to solve the integral equation (2.25), and the wavefield associated with wavenumbers over some threshold is truncated. Thus, too small a threshold may lead to an incomplete description of the total wavefield. Noting that

\[
    k_{\text{max}} = N_x \Delta k = \frac{N_x}{2} \frac{2\pi}{N_x \Delta x} = \frac{\pi}{\Delta x^2},
\]

we then take larger wavenumbers into the calculations of Fig. 10 by making \( \Delta x \) smaller. However, the PR results in Fig. 10 show fictitious oscillation caused by numerical instability for large wavenumbers corresponding to evanescent waves, while the IE results are stable and show quite similar shapes to those in Fig. 9.

We next show results for more complex structures having several irregular interfaces. In the structure of Fig. 11, the basin of Fig. 8 is divided into three layers, whose physical

Figure 9. The \( z \)-component response of a uniform sedimentary basin to a vertically incident \( P \)-wave on the free surface. The basin shown on the upper right side has a symmetric and smooth shape.

Figure 10. The same as in Fig. 9, but the displayed seismograms are calculated including larger wavenumbers than those in Fig. 9.
calculated $y$-component surface displacements with the same incident wavelet as in Fig. 8, and again found that the IE and PR results displayed in Fig. 11 agree well. We can observe Love waves generated at the basin edges, but their amplitudes are rapidly reduced at each reflection. Thus there are no major arrivals after 100 s, and so time delays are not distinct in the FD result. However, a numerical artifact appears in the FD solution at around 80 s, especially in the seismogram at a distance of 40 km. Since the addition rules (3.29) involve a number of manipulations of supermatrices, the speed advantage is reduced from 25 per cent for the single-interface case in Fig. 8 to 15 per cent for this three-interface case. On the other hand, the memory advantage is a little improved from 18 to 19 per cent, because a greater number of samplings is required than the single-interface case.

In Fig. 12, a new interface, which is represented by (4.8) with $D = 0$ km, $C = 0.5$ km and $w = 10$ km, divides the basin of Fig. 9, and an intermediate layer ($\alpha = 2.9$ km s$^{-1}$, $\beta = 1.6$ km s$^{-1}$, $\rho = 2.4$ g cm$^{-3}$) is introduced. The figure displays vertical surface displacements calculated with the same incident wavelet as in Fig. 9. Similarly to the previous $SH$ example in Fig. 11, the IE and PR results agree well, and the Rayleigh waves generated at the edge suddenly decay on the basin slope between distances of 1.6 and 2.4 km. However, the seismograms for this double-interface case include more high-frequency components than those for the single-interface case in Fig. 9. Both the speed and memory advantages are reduced from 45 and 29 per cent in Fig. 9 to 38 and 20 per cent, respectively.

Our final example is an $SH$ excitation problem including a transverse line force, for which we have to use the full

![Figure 11](image1.png)

**Figure 11.** The $y$-component response of a layered sedimentary basin to a vertically incident $SH$-wave on the free surface. The basin shown on the upper right side has three layers separated by asymmetric and rough interfaces.

parameters ($\beta, \rho$) are (0.7 km, 2.0 g cm$^{-3}$), (0.9, 2.1) and (1.5, 2.4), respectively. The shapes of the superficial interfaces are also represented by (4.6) with $D = 0$ or 0.5 km, $C = 1$ or 3 km, $w_1 = 20$ km and $w_2 = 30$ km. We

![Figure 12](image2.png)

**Figure 12.** The $z$-component response of a layer sedimentary basin to a vertically incident $P$-wave on the free surface. The basin shown on the upper right side has two layers separated by a symmetric and smooth interface.

![Figure 13](image3.png)

**Figure 13.** The $y$-component seismograms due to a line force buried in a uniform sedimentary layer at a depth of 0.5 km. The layer shown on the upper right side is bounded by a symmetric and smooth interface. All the traces are reduced with the velocity of the half-space. The arrows indicate the arrival of a head wave propagating along the interface.
Table 2. Comparison of computation efficiency.

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<th>Method</th>
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<th>adv.</th>
<th>Memory (Basic+Ext.)</th>
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IE : Invariant Embedding  
PR : Propagator  
FD : Finite Difference  
CPU time : total computation time  
VPU : computation time on the vector processor unit  
adv. : (PR - IE) / PR × 100

expression of (4.1) for the surface displacement instead of (4.5). The line force is set at 5 km depth in a low-velocity layer ($\beta = 2.0 \text{ km s}^{-1}, \rho = 2.5 \text{ g cm}^{-3}$) lying above a half-space ($\beta = 3.5 \text{ km s}^{-1}, \rho = 2.8 \text{ g cm}^{-3}$). The interface is represented by (4.8) with $D = 5.5 \text{ km}$, $C = -2.5 \text{ km}$ and $w = 50 \text{ km}$. This structure is almost the same as in the test case introduced by Koketsu (1987a), though he wrongly presented the parameters of the interface shape as $D = 3 \text{ km}$ and $C = 2.5 \text{ km}$. However, the S-wave velocity of the half-space is reduced slightly from 3.6 km s$^{-1}$ to avoid a mysterious interruption of the execution of the FD codes. In order to demonstrate numerical stability of our new approach for large wavenumbers, we adopt a smaller value for $Ax$ than Koketsu (1987a). (4.7) is used as a source time function with $t_p = 2 \text{ s}$ and $t_s = 1.83 \text{ s}$.

The surface seismograms are displayed in Fig. 13 with a reduction velocity of 3.5 km s$^{-1}$. The IE result agrees well to Koketsu's (1987a) one as well as to the FD seismograms, while the PR result is contaminated by a long-period noise due to numerical instability in evanescence. A head wave arrives at the times indicated by arrows, but its amplitude is much larger than in the IE seismograms. The reason for this may be that numerical parameters using the FD calculation are not quite appropriate for waves generated by an interface. The speed and memory advantages of the IE for this example are as good as in the $P$-$SV$ single-interface problem (Table 2).

5 DISCUSSION

We have already shown the merits of the new invariant embedding approach in the previous section, and so we now discuss its limitations.

Firstly, the addition rules for the reflection and transmission supermatrices (3.29) require a supermatrix inversion as well as a number of matrix multiplications, which are more complicated than the single supermatrix multiplication needed for the propagator technique. The extra computation is such that the IE method may lose its speed advantage if a medium has more than ten interfaces or so. However, it should hold the memory advantage even in such a case.

Secondly, we have had to impose the Rayleigh ansatz on every interface, while the propagator approach applies the ansatz only to the lowermost interface. The propagator treatment for the interfacial boundary condition is exact except for discretizing and truncation errors, and only the radiation conditions are based on the Rayleigh ansatz. Koketsu (1987a) suggested that this requirement can be avoided by introducing a dummy layer, but from (3.31) we know a uniform layer causes just a phase shift, and so his suggestion is not correct. In the invariant embedding formulation the ansatz is unavoidable on every interface.

However, we have shown good agreement for $SH$-waves between the invariant embedding results and those by the finite difference method, which does not involve the Rayleigh ansatz. Thus any error due to the application of the ansatz at each interface is small.

This Rayleigh ansatz error is potentially serious in theory, but truncation errors also raise significant problems. If we include large wavenumbers in order to avoid the truncation errors, the numerical instability problems are more severe in the propagator technique. As a result the propagator computation may well break down before the cumulative effects of the Rayleigh ansatz error have begun to influence the invariant embedding results. Similar problems were discussed by Axilrod & Ferguson (1990).

If we would like to avoid the Rayleigh ansatz error completely, we have to adopt the Green's function as a trial function instead of a plane wave. Since this results in an extremely long CPU time (see Axilrod & Ferguson 1990), it is preferable that we use the Green's functions only at steep interfaces, and plane waves are still taken at other moderate interfaces. This hybrid approach will soon appear in Takenaka, Koketsu & Kennett (1991, in preparation).

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