FAST-TRACK PAPER

A 2.5-D time-domain elastodynamic equation for a general anisotropic medium

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SUMMARY

In order to provide a quantitative analysis of real seismic records from complex regions we need to be able to calculate the wavefields in three dimensions. However, full 3-D modelling of seismic-wave propagation is still computationally intensive. An economical approach to the modelling of seismic-wave propagation which includes many important aspects of the propagation process is to examine the 3-D response of a model where the material parameters vary in two dimensions. Such a configuration, in which a 3-D wavefield is calculated for a 2-D medium, is called the '2.5-D problem'. Recently, Takenaka & Kennett (1996) proposed a 2.5-D time-domain elastodynamic equation for seismic wavefields in models with a 2-D variation in structure but obliquely incident plane waves in the absence of source. This approach is useful even for non-plane waves. In the presence of source a new 2.5-D elastodynamic equation for general anisotropic media can be derived in the time domain based on the Radon transform over slowness in the direction with constant medium properties. The approach can also be formulated in terms of velocity-stress, a representation which is well suited to the use of numerical techniques for 2-D time-domain problems such as velocity-stress finite-difference or velocity-stress pseudospectral techniques.

Key words: anisotropy, elastic-wave theory, seismic modelling, seismic waves, synthetic seismograms.

1 INTRODUCTION

Full 3-D modelling of seismic-wave propagation is still computationally intensive. Nevertheless, in order to provide a quantitative analysis of real seismic records from complex regions we need to be able to calculate the 3-D wavefields. Recently, as a compromise between realism and computational efficiency, two-and-a-half-dimensional (2.5-D) methods for calculating 3-D elastic wavefields in media varying in two dimensions have been developed. Such 2.5-D methods are an economical approach for calculating 3-D wavefields, and require a storage only slightly larger than those of the corresponding 2-D calculations.

Bleistein (1986) developed the ray-theoretical implications of 2.5-D modelling for acoustic problems. Luco, Wong & De Barros (1990) proposed a formulation for a 2.5-D indirect boundary method using Green's functions for a harmonic moving point force in order to obtain the 3-D response of an infinitely long canyon, in a layered half-space, for obliquely incident plane elastic waves. Pedersen, Sánchez-Sesma & Campillo (1994) also presented a 2.5-D indirect boundary element method based on moving Green's functions to investigate 3-D scattering of plane elastic waves by 2-D topographies. Takenaka, Kennett & Fujiwara (1996) have developed the 2.5-D discrete wavenumber–boundary integral equation method to study the interaction of the seismic wavefield excited by a point source with 2-D irregular topography. Randall (1991) developed a 2.5-D velocity–stress finite-difference technique for the calculation of multipole waveforms in non-axisymmetric boreholes and formations. Okamoto (1994) also presented a 2.5-D velocity–stress finite-difference technique, coupled with the reciprocal theorem, to calculate the teleseismic waveforms of subduction earthquakes. Furumura & Takenaka (1996) have developed an efficient 2.5-D formulation for the pseudospectral method for point-source excitation. The methods of Randall (1991), Okamoto (1994) and Furumura & Takenaka (1996) solve the elastodynamic equations in the time domain, while
the methods of Luco et al. (1990), Pedersen et al. (1994) and Takenaka et al. (1996) solve the equations in the frequency domain.

Randall (1991), Okamoto (1994) and Furumura & Takenaka (1996) applied a spatial Fourier transform to the 3-D time-domain elastodynamic equation in the direction along which the material parameters are constant, to recast the equations in a mixed coordinate–wavenumber domain. In this domain, 2.5-D problems can be solved using independent sets of 2-D equations for each wavenumber, using numerical techniques for 2-D time-domain problems such as the finite-difference method or the pseudospectral method followed by inverse Fourier transformation over wavenumber. In the approach of Okamoto (1994) the field quantities such as particle velocity and stress components are all complex-valued, while Randall (1991) and Furumura & Takenaka (1996) showed that these field quantities can be reduced to a real-valued case for symmetric or anti-symmetric sources with respect to the out-of-plane coordinate, along which the medium properties are constant.

In the present paper we propose a new 2.5-D time-domain elastodynamic equation for general anisotropic media which is based on the Radon transform instead of the Fourier transform. This equation is the extension of the 2.5-D elastodynamic equations for plane-wave incidence in an isotropic medium, derived by Takenaka & Kennett (1996), to excitation by a point source in a general anisotropic medium. The new equation is solved in a mixed coordinate–slowness domain. The solutions are then transformed to the spatial domain by integration (stacking) over slowness, which is equivalent to the inverse Radon transform. This equation is the extension of the elastodynamic equation for general anisotropic media which is based on the Radon transform instead of the Fourier transform, and can be recast as

\[ \rho \ddot{u_j} = \rho \dot{f}_j + \mathbf{f}, \quad i = 1, 2, 3. \]

(1)

For a general anisotropic medium, we introduce the set of stress components are all real-valued even for sources without spatial symmetry, and have employed the notation for derivatives: \( \partial_i = \partial / \partial x_i \), and \( \partial_j = \partial / \partial x_j \). The stress and displacement components are related by the 3-D generalized Hooke’s law through the elastic modulus tensor \( c_{ijkl} = c_{ijkl}(x_1, x_2, x_3) \) as follows:

\[ \tau_{ij} = c_{ijkl} \ddot{u}_j. \]

(2)

The elastodynamic equation (1) and the stress–displacement relation (2) can be recast as

\[ \rho \ddot{\mathbf{u}} = \rho \dot{\mathbf{f}} + \mathbf{f}, \]

(3)

and the traction vector \( \tau_r \) for normals along the coordinate axes,

\[ \tau_r = C_{ij} \ddot{u}_j, \quad r = 1, 2, 3. \]

(4)

We assume the medium is invariant in the \( x_3 \)-direction throughout the rest of this paper, so that the material properties take the form

\[ C_{ij} = C_{ij}(x_1, x_3). \]

(7)

On performing a Fourier transform of the 3-D equations (5) and (6) with respect to \( t \) and \( x_3 \), we obtain the following 2.5-D elastodynamic equation in the frequency–wavenumber domain:

\[ -\rho o^2 \ddot{\mathbf{u}} = \ddot{\mathbf{f}} + \mathbf{f}, \]

(8)

and the stress–displacement relations

\[ \tau_r = C_{ij} \frac{d}{dx_j} \mathbf{u} = -ik_r \tau_r \mathbf{u} + C_{ij} \frac{d}{dx_j} \mathbf{u}. \]

(9)

Here we have used the \( x_3 \)-invariance of the medium, i.e. eq. (7), and have employed the notation

\[ g(x_1, k_2, x_3, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx_3 \exp(\pm ik_2 x_3) \int_{-\infty}^{\infty} dt \exp(-i\omega t) g(x_1, x_2, x_3, t), \]

(10)

for the transform to the frequency–wavenumber domain.

2 DERIVATION VIA THE EQUATION IN THE FREQUENCY–WAVENUMBER DOMAIN

In this section, we derive a 2.5-D elastodynamic equation in the time domain for the situation of source excitation in a general anisotropic medium, from the 2.5-D equation in the frequency–wavenumber domain. The derivation process is similar to that given by Takenaka & Kennett (1996) to obtain the source-free 2.5-D elastodynamic equation in the time domain for an incident plane wave in an isotropic medium. Throughout this paper, we employ a Cartesian coordinate system \([x_1, x_2, x_3]\), where \( x_1 \) and \( x_2 \) are the horizontal coordinates and \( x_3 \) is the vertical one.

For a linear elastic medium, the 3-D time-domain elastodynamic equation with source terms is given by

\[ \rho \ddot{u}_i = \rho \dot{f}_i + \mathbf{f}, \quad i = 1, 2, 3, \]

(11)

where \( u_i = (u_i) = u_i(x_1, x_2, x_3, t) \) are the displacements at a point \((x_1, x_2, x_3)\) at time \( t \), the stress components are \( \tau_{ij} = \tau_{ij}(x_1, x_2, x_3, t) \), and \( f_i = (f_i) = f_i(x_1, x_2, x_3, t) \) are the body forces. The density \( \rho = \rho(x_1, x_2, x_3) \), and we have used the convention of summation over repeated suffices, and a contracted notation for derivatives: \( \partial_i = \partial / \partial x_i \), and \( \partial_j = \partial / \partial x_j \). The stress and displacement components are related by the 3-D generalized Hooke’s law through the elastic modulus tensor \( c_{ijkl} = c_{ijkl}(x_1, x_2, x_3) \) as follows:

\[ \tau_{ij} = c_{ijkl} \ddot{u}_j. \]

(2)

Here we have used the \( x_3 \)-invariance of the medium, i.e. eq. (7), and have employed the notation for derivatives: \( \partial_i = \partial / \partial x_i \), and \( \partial_j = \partial / \partial x_j \). The stress and displacement components are related by the 3-D generalized Hooke’s law through the elastic modulus tensor \( c_{ijkl} = c_{ijkl}(x_1, x_2, x_3) \) as follows:

\[ \tau_{ij} = c_{ijkl} \ddot{u}_j. \]

(2)

For a general anisotropic medium, we introduce the set of elastic modulus matrices \( c_{ij} \) (Woodhouse 1974) such that

\[ (C_{ij})_{kl} = c_{ijkl}, \]

(3)

and the traction vector \( \tau_r \) for normals along the coordinate axes,

\[ \tau_r = C_{ij} \ddot{u}_j, \quad r = 1, 2, 3. \]

(4)

We assume the medium is invariant in the \( x_3 \)-direction throughout the rest of this paper, so that the material properties take the form

\[ C_{ij} = C_{ij}(x_1, x_3). \]

(7)

On performing a Fourier transform of the 3-D equations (5) and (6) with respect to \( t \) and \( x_3 \), we obtain the following 2.5-D elastodynamic equation in the frequency–wavenumber domain:

\[ -\rho o^2 \ddot{\mathbf{u}} = \ddot{\mathbf{f}} + \mathbf{f}, \]

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and the stress–displacement relations

\[ \tau_r = C_{ij} \frac{d}{dx_j} \mathbf{u} = -ik_r \tau_r \mathbf{u} + C_{ij} \frac{d}{dx_j} \mathbf{u}. \]

(9)

Here we have used the \( x_3 \)-invariance of the medium, i.e. eq. (7), and have employed the notation

\[ g(x_1, k_2, x_3, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx_3 \exp(\pm ik_2 x_3) \int_{-\infty}^{\infty} dt \exp(-i\omega t) g(x_1, x_2, x_3, t), \]

(10)

for the transform to the frequency–wavenumber domain.

The inverse transform of the double Fourier transform (10) is

\[ g(x_1, x_2, x_3, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \exp\left(\pm iw t\right) \int_{-\infty}^{\infty} dk_2 \exp(-ik_2 x_2) \left[ I(x_1, k_2, x_3, \omega) \right], \]

(11)

Changing the order of the integration, and inserting the following relation between the wavenumber \( k_2 \) and the slowness \( p_2 \),

\[ k_2 = \omega p_2, \]

(12)

eq. (11) leads to

\[ g(x_1, x_2, x_3, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dw \exp\left(\pm i\omega t - p_2 x_2\right) \left| I(x_1, \omega p_2, x_3, \omega) \right|, \]

(13)
The arguments of the field quantities in eqs (17) and (18) are all \((x_1, p_2, x_3, t - p_2 x_2)\). Note that these equations include no derivatives with respect to \(x_2\), and that all the quantities in these equations are real-valued.

### 3 VELOCITY–STRESS FORMULATION

The set of equations (17) and (18) derived in the previous section represents the elasticodynamic response of a 2.5-D anisotropic medium in the presence of source and incorporates Hooke’s law. In this section we show the velocity–stress formulation of this set.

Differentiation of eq. (18) with respect to time gives

\[
\partial_t \xi_t = C_{11} \partial_t \xi \xi - p_2 C_{12} \partial_{x_2} \xi - p_2 C_{32} \partial_{x_3} \xi + \hat{f},
\]

in terms of the particle velocity \(\xi\). On substituting (19) into (17) and extracting the first derivative of the particle velocity vector, we have

\[
\rho \partial_t \xi = \partial_t \xi \xi - p_2 C_{21} \partial_{x_1} \xi + C_{22} \partial_{x_2} \xi + (\partial_{x_1} \partial_{x_1} + \partial_{x_2} \partial_{x_2} + \hat{f}),
\]

where \(I\) is the three-by-three identity matrix. Pre-multiplying (20) by the inverse of the inertia matrix \([\rho I - p_2^2 C_{22}]\), we obtain

\[
\partial_t \xi = -p_2 [\rho I - p_2^2 C_{22}]^{-1}(C_{21} \partial_{x_1} \xi - C_{22} \partial_{x_2} \xi) + [\rho I - p_2^2 C_{22}]^{-1}(\partial_{x_1} \xi + \partial_{x_2} \xi + \hat{f}).
\]

The time derivative \(\partial_t \xi\) can now be replaced in (19) to give the more convenient evolution equation

\[
\partial_t \xi = [C_{11} + p_2^2 C_{22} [\rho I - p_2^2 C_{22}]^{-1} C_{21}] \partial_{x_1} \xi + [C_{22} + p_2^2 C_{32} [\rho I - p_2^2 C_{22}]^{-1} C_{32}] \partial_{x_2} \xi - p_2 C_{22} [\rho I - p_2^2 C_{22}]^{-1}(\partial_{x_1} \xi + \partial_{x_2} \xi + \hat{f}).
\]

The set of equations (21) and (22) represents a velocity–stress formulation of the 2.5-D elastodynamic equation. Note that the actual number of the equations in this formulation is nine, since the stress tensor is symmetric. This formulation is useful for numerical methods such as velocity–stress finite-difference or velocity–stress pseudospectral techniques.

Eqs (21) and (22) as well as eqs (17) and (18) are represented for general anisotropy. For transverse isotropy with a vertical symmetry axis, the corresponding equations can be derived from the elastic modulus matrices \(C_{ij}\):

\[
C_{11} = \text{diag}(A, N, L), \quad C_{22} = \text{diag}(N, A, L), \quad C_{33} = \text{diag}(L, L, C),
\]

\[
C_{12} = C_{21}^T = \begin{bmatrix} 0 & H & 0 \\ 0 & 0 & F \\ 0 & L & 0 \end{bmatrix}, \quad C_{13} = C_{31}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ L & 0 & 0 \end{bmatrix},
\]

and the inertia matrix \([\rho I - p_2^2 C_{22}]\) is reduced to

\[
\rho I - p_2^2 C_{22} = \text{diag}(\rho - p_2^2 N, \rho - p_2^2 A, \rho - p_2^2 L).
\]

For the further restriction to an isotropic medium,

\[
A = C = \lambda + 2 \mu, \quad F = H = \lambda, \quad N = L = \mu,
\]

\[
[\rho I - p_2^2 C_{22}] = \text{diag}(\rho, \rho, \rho)
\]

where \(\lambda\) and \(\mu\) are the Lamé constants, and

\[
\rho_s \equiv \rho - p_2^2 (\lambda + 2 \mu) = \rho (1 - \alpha^2 p_2^2),
\]

with \(P\)-wave velocity \(\alpha\) and \(S\)-wave velocity \(\beta\).

### 4 DISCUSSION

In the preceding sections, we have presented the 2.5-D elastodynamic equation for source excitation as well as the 2.5-D equivalent of Hooke’s law, and a velocity–stress formulation of the equations. For 2.5-D modelling of a 3-D wavefield excited by a source using these equations, we need first to transform the spatial and temporal behaviour of the source to the time-slowness domain as defined by eq. (14). This transform can also be performed by the following relation:

\[
\mathcal{G}(x_1, p_2, x_3, t) = -\frac{1}{2\pi} \mathcal{R}\left[ \frac{d}{dt} \mathcal{H}[g] \right] = -\frac{1}{2\pi} \frac{d}{dt} \mathcal{H}[\mathcal{R}[g]],
\]

where \(\mathcal{R}[\_]\) and \(\mathcal{H}[\_]\) are the Radon and Hilbert transform operators, respectively, defined as

\[
\mathcal{R}[g](x_1, p_2, x_3) = \int_{-\infty}^{\infty} dx_2 g(x_1, x_2, x_3, t + p_2 x_2),
\]

\[
\mathcal{H}[g](x_1, x_2, x_3) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{g(x_1, x_2, x_3, t)}{t - t'}
\]

(see e.g. Chapman 1978; McCowan & Brysk 1989).

We recall that the arguments of the field quantities in eqs (17), (18), (21) and (22) are all \((x_1, p_2, x_3, t - p_2 x_2)\). When we solve the set of equations (17) and (18), or eqs (21) and (22),
we can set $x_2 = 0$, so that these equations are reduced to the 2-D case. For a fixed value of the slowness $p_2$, eqs (17), (18), (21) and (22) then depend on only two space coordinates, i.e. $x_1$ and $x_3$. For each value of $p_2$, these equations can therefore be solved as independent 2-D equations. The invariance of the medium in the $x_2$-direction means that there is no coupling between different $p_2$ components, while for full 3-D problems there would be coupling between different $p_2$ slowness components. Once the set of equations (17) and (18), or eqs (21) and (22) have been solved for $x_2 = 0$, we can deduce the space-domain displacement $u(x_1, x_2, x_3, t)$ [or velocity $\dot{u}(x_1, x_2, x_3, t)$] at any $x_2$ from the solutions at $x_2 = 0$, $\dot{u}(x_1, p_2, x_3, t)$ [or $\dot{\hat{u}}(x_1, x_2, x_3, t)$], by a slant stack over slowness $p_2$:

$$u(x_1, x_2, x_3, t) = \int_{-\infty}^{\infty} dp_2 \dot{u}(x_1, p_2, x_3, t - p_2 x_2),$$

(31)

or

$$\dot{u}(x_1, x_2, x_3, t) = \int_{-\infty}^{\infty} dp_2 \dot{\hat{u}}(x_1, p_2, x_3, t - p_2 x_2),$$

(32)

(see eq. 13).

5 CONCLUSION

We have derived a new 2.5-D time-domain elastodynamic equation for a general anisotropic medium which is based on the Radon transform. This is an alternative to the equations based on the Fourier transform (e.g. Randall 1991; Furumura & Takenaka 1996). The new equation is solved in a mixed coordinate–slowness domain. The solutions are then transformed to the spatial domain by the inverse Radon transform, which must be performed numerically for the point-source case. All quantities in these calculations are real-valued, which represents a significant advantage for the new form of equation. The real-number computations require less memory and computation time than the alternative approach to the general point-source case using a Fourier transform over wavenumbers. Experience with slowness integration for reflectivity methods suggests that the evaluation of the inverse Radon transform would require a higher number of samples than for the Fourier transform, but not sufficient to outweigh the computational advantages of the slowness approach.

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