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# Signal Parameter Estimation for Sparse Arrays

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**Abstract** For arrays with a small number of elements, such as those deployed in infrasound detection, the theoretical pattern of beam power associated with the incidence of a plane wave shows a broad main beam with strong side lobes. A novel approach to the estimation of the parameters of the incoming wave field is proposed based on the exploitation of the full beam pattern for the array. The pattern of beam power over a limited mask of beam points in slowness space is compared with the theoretical predictions over a set of narrow frequency bands and the set of signal parameters that give the best fit determined by an inversion using a Neighbourhood Algorithm (NA). The angle of incidence and azimuth provide a convenient parameter set since they can be used to include corrections for elevation differences between the sensors in the process of beamforming. Because the theoretical beam pattern depends only on the array geometry and frequency band, it can be computed on a dense grid and interpolation carried out to provide the beam power across the beam mask for a specified set of angular parameters. The NA inversion with an $L_{1.3}$ measure of misfit in beam power is able to achieve good definition of the minimum misfit with only 120 trials, and the level of fit itself provides a good measure of the validity of the model of a single dominant plane wave. The NA approach is supplemented by a contracting grid method using a set of self-similar grid masks that can rapidly locate the position of maximum beam power. The two methods provide a valuable tool for real-time analysis, with both independent estimates of the angular parameters and a measure of the quality of the results.

## Introduction

A variety of techniques have been developed for estimating the slowness characteristics of seismic signals across arrays. The analysis is based on a model of an incoming plane wave, and the array designs contain enough stations to provide amplification of the main lobe (peaked at the desired slowness parameters) relative to the subsidiary side bands (see, e.g., Abrahamson and Bolt, 1987; Schweitzer *et al.*, 2003).

With sufficient spatial distribution of sensors a direct *f*-*k* Fourier approach can be used; alternatively a dense set of beams sampling slowness space can be used to locate the maximum beam power. Various devices, such as the use of semblance or phase stacks, have been introduced to concentrate attention on the coherent signal across the array (cf. Kennett, 2000). Nonlinear semblance- or phase-weighted stacks have good signal-enhancement properties. A similar result can be achieved with the *N*th root method (Muirhead, 1968).

However, when there are only a few sensors, the separation in the amplitude of the main and side lobes is modest at any particular frequency. The relatively broad peak for the main lobe can be sharpened by using broadband signals, but complications can arise if the character of the wave field varies with frequency. The problem of beamforming is particularly acute for the four- or eight-element infrasound arrays currently being installed as part of the International Monitoring System (IMS) for the Comprehensive Nuclear-Test-Ban Treaty. Hitherto attention has been concentrated on the enhancement of the main lobe, but the whole array response bears the imprint of the incident slowness parameters and can be exploited to extract these parameters.

A successful algorithm for slowness estimation has been developed based on beamforming for a limited number of points on a slowness grid, coupled with an inversion to minimize the misfit between the observed and theoretical distributions of beam power across this slowness mask. The inversion exploits the parameter space exploration of the neighborhood algorithm (NA) (Sambridge, 1999) to home in on the region of best fit. Although the misfit surfaces are very complex with multiple minima, good results can be achieved with a surprisingly small number of slowness samples. The theoretical beam power can be precalculated and simply needs to be shifted to each target horizontal slowness pair and then interpolated to the sampling mask. The computational expense of each trial is therefore very low. Multiple frequency bands can be handled in real time to provide a full characterization of the acoustic wave field.

The inversion procedure depends on the model of a single incident plane wave at the array, and the misfit levels provide a measure of the validity of this assumption. It is therefore useful to combine the inversion approach with a direct estimate of maximizing beam power. This can be achieved by using a contracting grid approach working from the same set of standard beams.

Where there is significant elevation difference between sensors, the beamforming needs to take account of the vertical component of slowness, which can readily be accommodated by working in terms of the angle of propagation to the vertical $\theta$ and the azimuth to north $\varphi$. The beams are phased to a reference plane and indexed in terms of the horizontal slowness.

## Array Beaming

The response of an array of $j$ elements at the positions $\mathbf{x}_j$ to an incident plane wave with slowness vector $\mathbf{p}$, at frequency $\omega$, can be characterized by the array response function as a function of slowness $\mathbf{s}$:

$$S(\mathbf{p},\ \mathbf{s},\ \omega) = \sum_j e^{i\omega(\mathbf{p}-\mathbf{s})\cdot\mathbf{x}_j}, \tag{1}$$

which depends only on the difference in slowness $\Delta\mathbf{s} = \mathbf{p} - \mathbf{s}$. Over a band of frequencies $(\omega_1,\ \omega_2)$ the theoretical beam power is thus

$$S(\mathbf{p},\ \mathbf{s}) = \int_{\omega_1}^{\omega_2} d\omega \left|\sum_j e^{i\omega\Delta\mathbf{s}\cdot\mathbf{x}_j}\right|^2. \tag{2}$$

The beam power pattern can thus be computed directly for a vertically incident wave for which $p_1 = p_2 = 0$ and then the whole pattern shifted to be centred at the required incident slowness $\mathbf{p}$.

With a set of observations $v_j(\mathbf{t})$ at the stations in the array, we can create linear beam estimates by a delay and sum technique:

$$B(\mathbf{s},\ t) = \sum_j \hat{v}_j(t - \mathbf{s}\cdot\mathbf{x}_j), \tag{3}$$

where $\hat{v}_j$ is a normalized trace. In the frequency domain

$$\bar{B}(\mathbf{s},\ \omega) = \sum_j \hat{v}_j(\omega)\ e^{i\omega\mathbf{s}\cdot\mathbf{x}_j}, \tag{4}$$

and we can construct a beam power estimate comparable to equation (2) as

$$\mathcal{B}(\mathbf{s}) = \int_{\omega_1}^{\omega_2} d\omega \left|\sum_j \hat{v}_j(\omega)e^{i\omega\mathbf{s}\cdot\mathbf{x}_j}\right|^2. \tag{5}$$

The inversion is based on comparing $\mathcal{B}(\mathbf{s})$ with $S(\mathbf{p},\mathbf{s})$ centred on a target slowness $\mathbf{p}$.

For significant elevation differences between sensors, the beamforming needs to take account of the vertical component of slowness, which can readily be accommodated by working in terms of the angle of propagation to the vertical $\theta$ and the azimuth to north $\varphi$. For a propagating acoustic wave, with local sound speed $v_s$,

$$s_z = \frac{\cos\theta}{v_s},\quad s_n = \frac{\sin\theta\cos\varphi}{v_s},\quad s_e = \frac{\sin\theta\cos\varphi}{v_s}. \tag{6}$$

The beams are phased to a reference plane $z = H$, with a correction for the height differences through the $s_z(H - z_j)$ contribution to the phase term at each site, and indexed in terms of the horizontal slowness pair or the angular parameters. This means that we need only employ two parameters to describe the incoming plane wave.

## Angular Parameter Estimation

We employ two different styles of approach to the extraction of the angular parameters for the incoming signal. The first is based on matching observed and theoretical patterns of beam power over a limited number of beams, using exploration of angular parameter space with the NA of Sambridge (1999). The second is based on just finding the maximum beam power in horizontal slowness space, as an indicator of the currently most important signal across the array.

We present examples based on the IS07 infrasound array at the Warramunga station in the Northern Territory of Australia operated by the Australian National University (Fig. 1). This eight-element array has been built as part of the IMS and comprises four low-frequency elements in a centered triangle and four high-frequency elements in a much smaller triangle with a near coincident center point. Each of the sensors is connected to a radiating pipe array designed to minimize the effects of wind noise; larger arrays are used for the low-frequency elements. Data from all the sensors are sent by digital telemetry to the central recording laboratory.

To optimize the application to real-time information, both of our methods for the estimation of angular parameters start from the same limited set of beam power estimates. A restricted beam mask is employed to create stacks of the incoming data for 12.8-sec intervals, corresponding to 256 time samples for the IMS infrasound arrays. A fast Fourier transform (FFT) is used to convert to the frequency domain. Stabilized beam power estimates are then made by averaging over the three discrete frequencies surrounding the required target frequency.

We have achieved good results with both an 81-element rectangular grid in horizontal slowness and the 85-beam mask illustrated in Figure 2, which provides relatively uni-

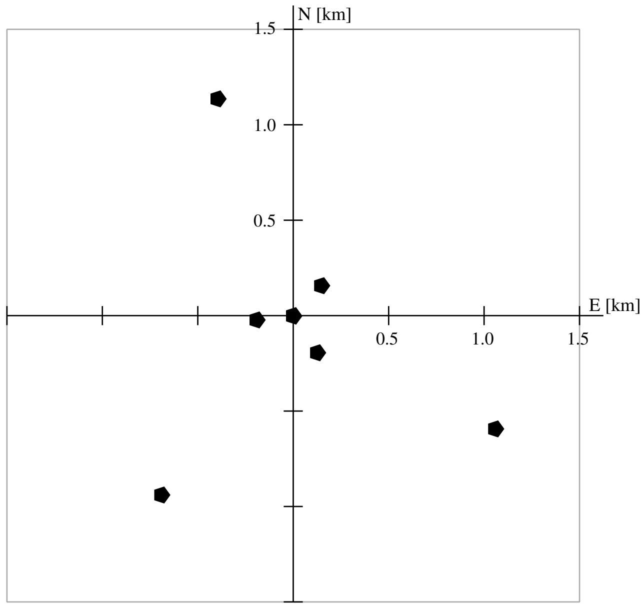


Figure 1. Geometry of the IS07 infrasound array in northern Australia used for the illustrations of angular parameter estimation.

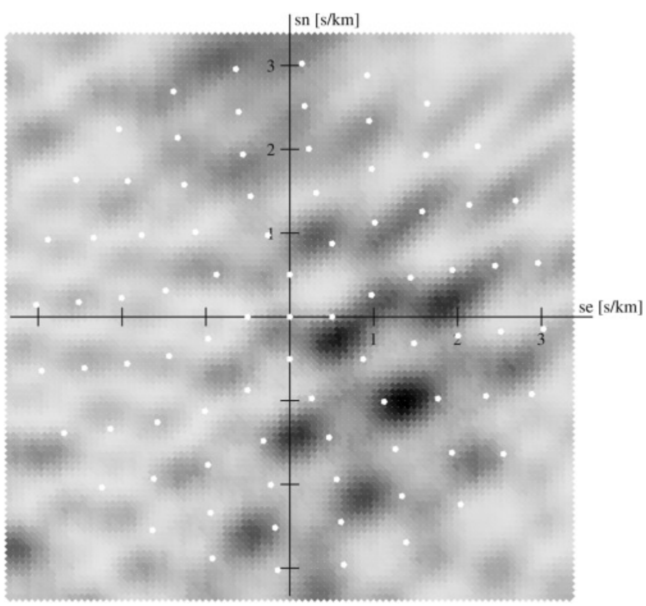


Figure 2. Beam power in horizontal slowness space (darker tones represent higher power) and configuration of the basis 85-beam mask; example for a narrow band around 0.875 Hz.

form sampling of horizontal slowness space for propagating acoustic waves so that each point is defined by real values of $\theta$, $\varphi$.

## Matching Beam Power Patterns

Our new method for the estimation of the angular properties of the incoming acoustic waves is based on the model of a single dominant plane wave over each time interval. The algorithm is based on comparing the patterns of beam power across a limited number of points in the horizontal slowness domain between stacks created from the observations and theoretical estimates. An inversion for the angular parameters $\theta$, $\varphi$ is undertaken to minimize the misfit between the observed and theoretical distributions of beam power across the beam mask. The inversion exploits the parameter space exploration of the NA (Sambridge, 1999) to home in on the region of best fit. Although the misfit surface is very complex with multiple minima, good results can be achieved with a surprisingly small number of slowness samples.

Because the theoretical beam power pattern for a given frequency band depends only on the geometry of the array, it can be precalculated. When required, the pattern needs to be shifted to each target horizontal slowness pair and then interpolated onto the sampling mask. The computational expense of each trial is therefore very low. We have used a $401 \times 401$ grid for the theoretical distribution for each narrow band in frequency for a slowness range of $(-0.6, 0.6)$ sec/km in both $\Delta s_n$, $\Delta s_e$, so that the target point can be placed anywhere in the domain of propagating acoustic waves. Bilinear interpolation is used to generate the appropriate beam power estimates at the required slowness values. Once calculated for a given array geometry and frequency band, this theoretical distribution can be reused with minimal cost.

Because the NA method does not evaluate any derivatives, we are able to work with any suitable measure of the misfit function between the true beam power evaluated at the 85 points of the standard beam mask and the corresponding estimates for the theoretical beam pattern centered on a target slowness $\mathbf{p}$. We have experimented with a variety of different $L_p$ measures of this misfit function of the form

$$\mathcal{M}_p(\mathbf{p}) = \left(\frac{1}{85} \sum_{i=1}^{85} \left| S(\mathbf{p}, \mathbf{s}_i) - \mathcal{B}(\mathbf{s}_i) \right|^p \right)^{1/p}, \qquad (7)$$

with both patterns normalized to the peak power found across the 85-beam mask for the appropriate narrow frequency band. With the use of a narrow band of frequencies we do not need to investigate the details of the frequency content of the signal and can cope with relatively rapid changes in signal amplitude by using a sequence of inversions for different center frequencies.

We have undertaken a range of tests using synthetic signals to investigate the dependence of the inversion for beam pattern on the choice of $p$ for the misfit function. For theoretical patterns without noise it is possible to use a standard $\mathcal{M}_2$ measure, but once noise is added and the single plane-wave model is thus contaminated, the parameter estimates are poor with $p = 2$. However, even with significant levels of noise it is possible to achieve good recovery of the signal parameters with $p < 1.5$. A very good compromise between robustness and sensitivity to outliers arising from extraneous noise comes with $p = 1.3$, and we will use this $\mathcal{M}_{1.3}$ measure in the subsequent examples.

The behavior of this $\mathcal{M}_{1.3}$ misfit function as a function of beam point in the horizontal slowness plane is shown in Figure 3 for the beam power pattern illustrated in Figure 2. The minimum misfit, indicated in black in Figure 3, is found close to the maximum beam power (see Figs. 2, 5) but the misfit surface is exceedingly complicated with multiple maxima and minima extending across the whole slowness domain.

Despite the complexity of the misfit surface, an NA inversion working in terms of the two angles is very effective. The procedure is based on a systematic exploration of parameter space. The procedure is started with an initial random distribution of samples, and the parameter space is then divided into Voronoi cells, which represent the zone that is closest to each sample point. Each of these cells is characterized by the misfit at the sample point. New samples are then distributed among a certain number of cells with least misfit. With the addition of the misfit information at these new points, the Voronoi cell pattern is reconstructed and the process repeated. All the prior information is used in each update of the representation of the behavior of the parameter space, and the method is thereby able to follow multiple minima until one emerges as the favored point. In a certain sense the NA approach can be regarded as multiple contracting grids working simultaneously.

Convergence of the NA scheme can be achieved with only 120 trial slowness points (30 initial and 10 iterations with nine new trials and the six best cells sampled). The neighborhood of the best fit is found quickly and then refined (see Fig. 4). The method works well with parametrization in terms of either horizontal slownesses or the angular parameters for a 2D array. However, because of the convenience in handling elevation differences across arrays, we favor the direct use of the angular parameters.

The success of this inversion approach to the determination of the parameters of the signal comes from the fact that we are exploiting the full set of information including the side lobes in the beam pattern. Tests with synthetic data indicate that little degradation in the results occurs until the level of noise exceeds one-third of the signal. The features of the array response pattern remain distinct even though some parts may be modified, and this is sufficient to achieve an effective match. The error in the parameter estimates increase quickly once the signal-to-noise ratio (SNR) is less than 2.

The NA method thus works well when the data conform to the basic assumption of a single dominant plane wave incident on the array. We have found that the size of the minimum misfit $\mathcal{M}_{1.3}$ found in the inversion process gives a good indication of the validity of the model for the signal; small misfit corresponds to a clear plane-wave signal, and larger misfits indicate that there is no single dominant version of the expected array pattern. This method fails, as would be expected, when there are multiple arrivals of comparable amplitude.

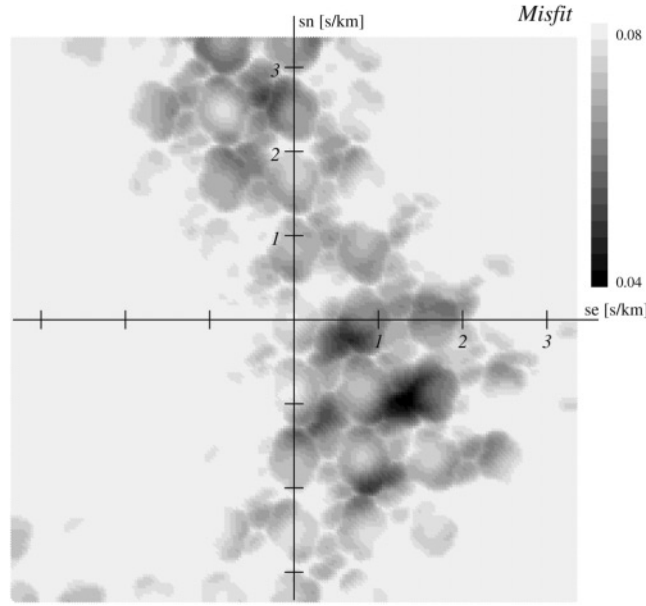


Figure 3. Misfit $M_{1.3}$ between beam power patterns over a fixed 85-beam mask as a function of slowness for the pattern shown in Figure 2.

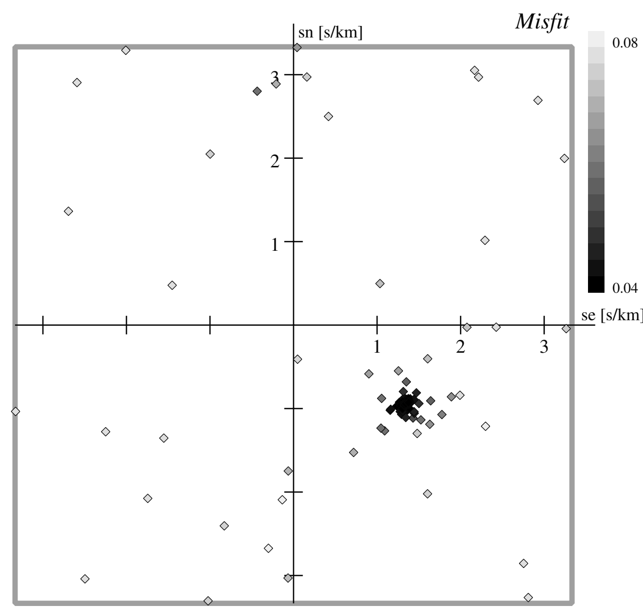


Figure 4. Convergence of NA inversion for the same case as in Figure 3, with the misfit $M_{1.3}$ in the same tones. The region of least misfit in the beam power patterns across the 85-beam mask is found quickly and is well controlled.

The quality of the recovered parameter estimates is sustained until the the SNR is less than 3; the properties of the best 10 slowness samples then provide an indication of the potential spread of the parameters. When the misfit level is high, the scatter in the parameters associated with comparable misfit is much larger and again can be used as an indication of the constraint on the parameter estimates.

### Maximum Beam Power by Contracting Grid

It is always possible to locate the maximum in the beam power by exhaustive search, for example, with a systematic grid in horizontal slowness space. For satisfactory results at 1 Hz, with a typical infrasound array configuration, a minimum sampling to achieve adequate definition would be $61 \times 61$ points (i.e., 3721 beams). An alternative strategy is to employ a preliminary systematic search to find the neighborhood of the global maximum and then use a nonlinear optimization procedure; in this case about $31 \times 31$ points (i.e., 961 beams) would be needed to achieve good definition of the search point, to which must be added the cost of the final nonlinear optimization.

An alternative approach that yields good results with relatively few beam evaluations is based on the strategy of using a contracting grid of beams centered about the current estimate of maximum beam power. Very effective results can be achieved by using a factor of 2 reduction in the radius of the beam pattern at each iteration. With the 85-beam mask, illustrated in Figure 2, in three iterations of the contracting grid scheme the maximum beam power can be located to within 1° in incidence angle and azimuth (see Fig. 5); this is well within the theoretical resolution limit for the eight-element IS07 array. The total number of beam evaluations is then just 340, with a high density of points around the actual maximum from which to estimate the likely variance of the angular parameters.

Because the self-similar beam pattern covers a significant region of slowness space at each iteration there is no requirement that the initial estimate is close to the final location, although it often will be. Significant migration of the contracting grids is also possible, when the basic grid has only sampled the flanks of the major features in the beam power pattern.

The utility of the contracting grid procedure is that with a relatively low number of beam evaluations it can give a very well defined estimate of the angular parameters corresponding to maximum beam power, irrespective of the nature of the input signal. This independent technique can therefore provide a useful control on the results from the beam power pattern inversion, which are strongly dependent on the model of the signal being employed.

## Application of the Beam Pattern Method to Continuous Data

The inversion procedure for matching the beam power pattern using NA has been adapted to work in a loop mode to handle a stream of real-time infrasound data. A fixed set of parameters is then employed with the same random seed for each time window. The role of this seed is to determine the sampling points used both in initializing the procedure

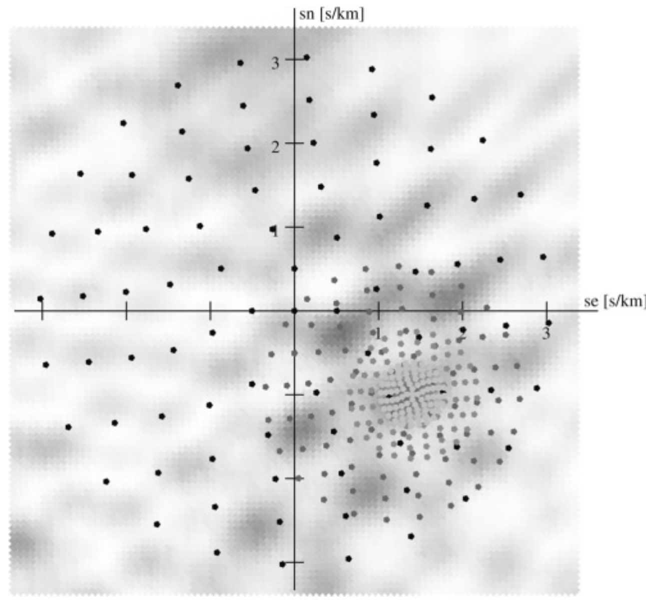


Figure 5. The progress of the contracting grid scheme in finding the maximum in beam power for the pattern shown in Figure 1. The successive self-similar grids are reduced in radius by a factor of 2 and are shown in different tones. By the third iteration a very tight control is achieved on the postion of the maximum beam power.

and during the subsequent refinement. Although, in principle, any random seed could be used if a large number of samples are employed, careful testing is needed to determine a satisfactory random number sequence when used with a limited number of samples.

The essence of our approach, as discussed earlier, is the matching of the beam power pattern for the observations over a narrow frequency band to the theoretical estimates based on a model of a single incident plane wave. We have used the inverse of the minimum misfit [$1/\mathcal{M}_{1,3}$ (min)] as a gauge of the reliability of the single plane-wave model and have been able to turn this into a measure of the quality of the local estimate of the signal parameters on a four-point scale.

For the continuous data the set of beams is formed over a 12.8-sec window, and then a 256-point FFT is used to extract beam power estimates averaged over three contiguous frequencies. The NA inversion is applied to the beam power pattern across the 85-beam mask for a number of narrowband frequency windows to determine the local estimates of the angular parameters accompanied by their quality estimates. The window is then moved up by 6.4 sec and the process repeated. The cost of the inversion step is minimized because the theoretical beam power pattern does not have to be recomputed and only interpolation is need for each new beam sample.

The process of continuous analysis is illustrated with an example of records from the IS07 array (Warramunga, Australia) into which a new signal segment with known properties is inserted (Fig. 6). This simulation allows investigation of the effect of the SNR, since the size of the inserted signal can be adjusted. The dominant power in the known signal lies below 0.6 Hz. This example is a composite of real data and a synthetic signal with specified properties so that we are able to test the recovery of a known result.

The results of the NA inversion are shown in Figure 7 for two narrow frequency bands. The upper panel shows the 0.5-Hz frequency band for which the SNR is about 3 for the segment with injected signal; the lower panel shows the 0.75-Hz frequency band for which the SNR falls below 1.5 in many places. In each case the presence of the injected signal over a substantial time interval can be picked out by the consistent estimate of the azimuth $\varphi$, accompanied by an increase in the assigned quality of the estimates. There is some scatter in the angle of incidence estimates $\theta$, but the parameters of the inserted signal are well recovered even with limited SNR. Intriguingly the angle of incidence values are more coherent for a segment of the higher frequency band.

On a SUN Ultra 5, the angular parameter estimation for each frequency band for 1 hr of data takes about 42 sec, including the 9-sec cost of establishing the initial detailed version of the theoretical beam power distribution for which we use a $401 \times 401$ grid. Once calculated for a given array geometry this information can be stored and reused with very low computational cost.

In applications to infrasound signals from a number of infrasound stations, we have found that the use of multiple frequency bands is highly desirable to allow some degree of tuning to signal characteristics. Because the comparison of the beam power pattern is made with theoretical results that assume full coherency across the arrays, significant variations in the signals at different stations will degrade the performance of the method even when the SNR is good.

Even in circumstances where there are no clear signals, the beam power pattern approach can indicate the presence of a new class of arrivals by the disruption of the pattern of parameter estimates associated with the background noise. In this case, however, the angular parameters in such an interval would need to be treated with caution.

## Discussion

The two approaches we have used to identify the angular parameters describing the incoming signal at a small-aperture array have a useful complementarity, and both are simple enough to be applied in real-time simultaneously at a number of arrays.

The inversion scheme provides a good estimate of the angular parameters when there is a clear plane-wave component to the incident field, but also can be used to assess the validity of this assumption. The parameters are those that provide the best match between the beam power determined from the observations and the expected array response for the frequency band.

The contracting grid approach, on the other hand, starts with the same basic beam mask in horizontal slowness space, but is aimed at finding the point of maximum beam power. This point would be expected to coincide with the position of the peak in the main lobe of the array pattern. Even for narrow bands in frequency, there can be some slight shift from the position of best pattern match for large signals.

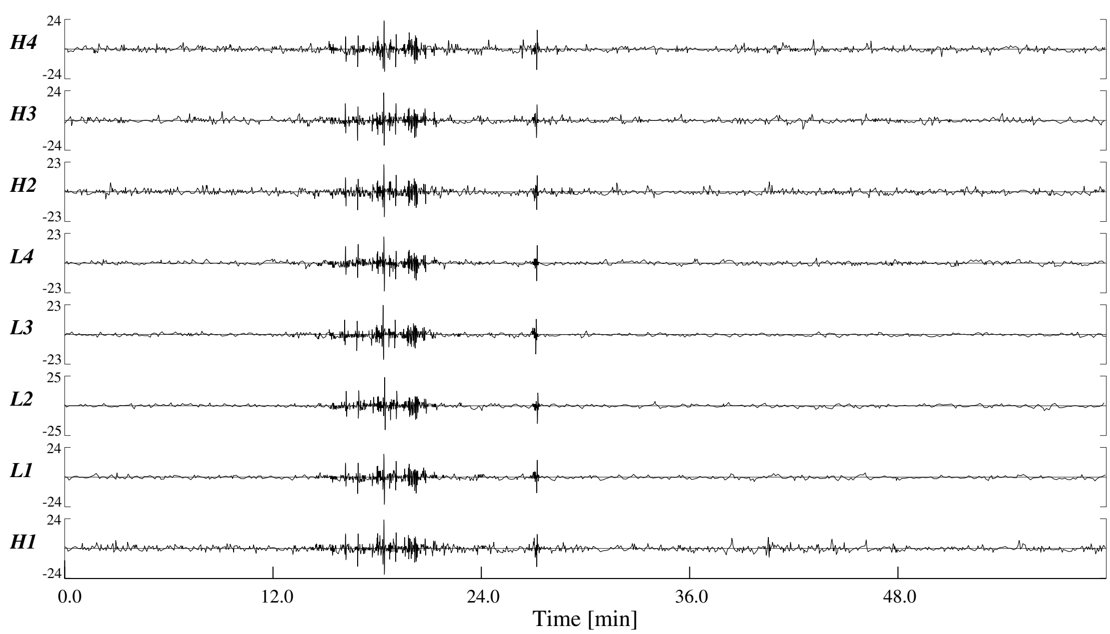


Figure 6. Segment of infrasound record from IS07 to which has been added a component with known parameters (incidence angle $\theta = 57°$, azimuth $\varphi = 128°$).

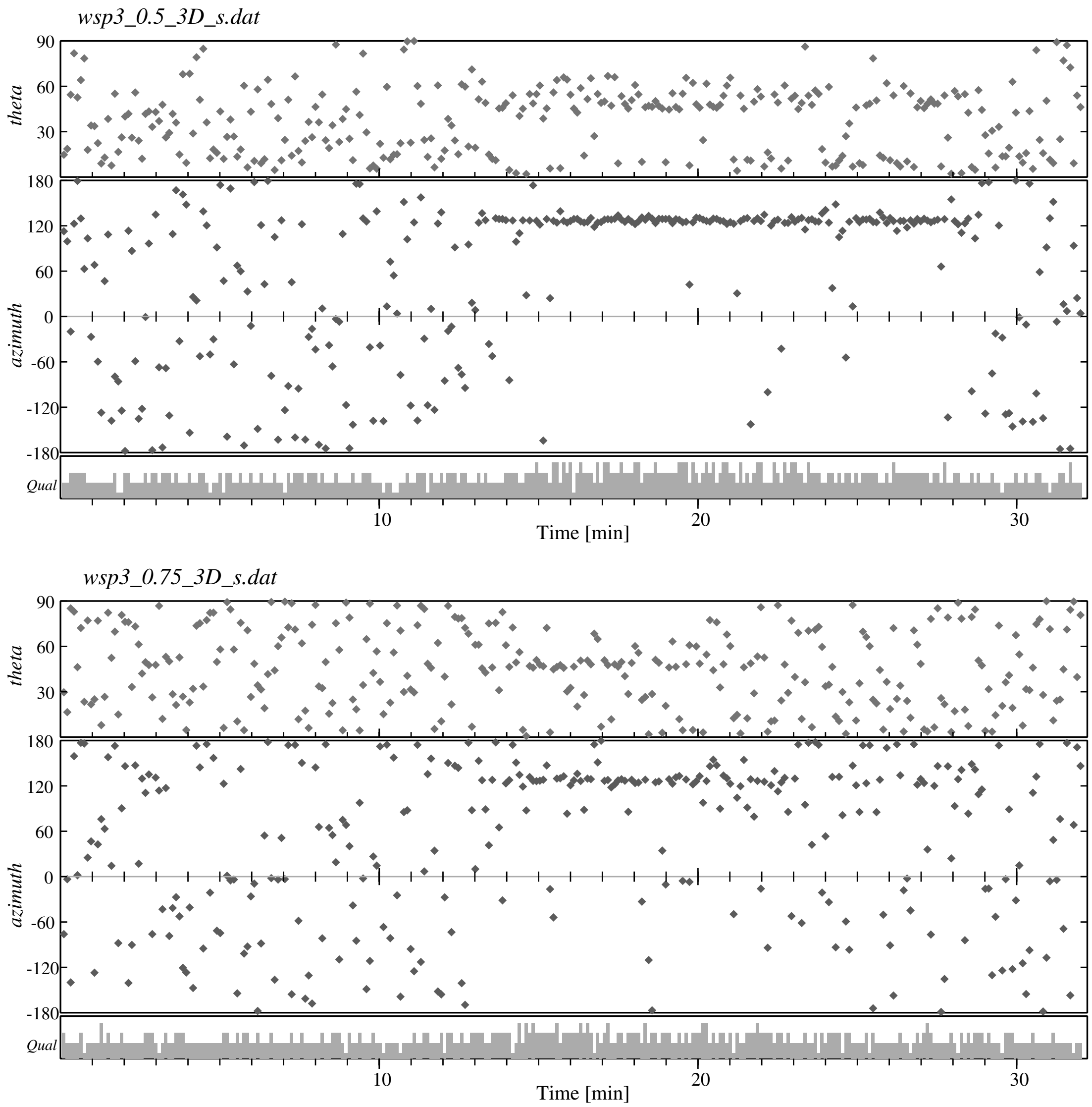


Figure 7. NA inversion for angular parameters for continuous data. The upper panel shows a narrow frequency band centered on 0.5 Hz, and the lower panel shows a band centered on 0.75 Hz, where the signal-to-noise ratio is much lower. In each case the segment of data where the known signal has been inserted is readily identifiable by the consistency in azimuth over a significant time interval. The azimuth estimate is very good in the 0.5 Hz band, but there is more variation in the angle of incidence, due to the influence of the background noise.

However, when the quality of the inversion procedure is high, as judged by the small misfit, we expect a close correspondence between the two methods.

Differences between the parameter estimates from the two techniques are indicative of the presence of multiple components in the acoustic wave field. The contracting grid scheme will then provide an estimate of the parameters for the largest component (although this can be corrupted if there is significant overlap between the beam patterns associated with different arrivals).

The inversion approach could be adapted to a model of two or more signals, with a parametrization including the relative strength of the arrivals, but the multidimensional search procedure to determine the angular component for the different components would undoubtedly require significantly more computation than for the single plane-wave model. Such analysis would be more suitable for off-line work than real-time monitoring.

Further material on the NA employed in the beam pattern inversion, including papers, graphics, and computer code, may be downloaded at http://rses.anu.edu.au/∼malcolm.

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