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The Ambiguity in Ray Perturbation Theory

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Ray perturbation theory is concerned with the change in ray paths and travel times due to changes in the slowness model or the end-point conditions of rays. Several different formulations of ray perturbation theory have been developed. Even for the same physical problem different perturbation equations have been derived. The reason for this is that ray perturbation theory contains a fundamental ambiguity. One can move a point along a curve without changing the shape of the curve. This means that the mapping from a reference curve to a perturbed curve is not uniquely defined, because one may associate a point on the reference curve with different points on the perturbed curve. The mapping that is used is usually defined implicitly by the choice of the coordinate system or the independent parameter. In this paper, a formalism is developed where one can specify explicitly the mapping from the reference curve to the perturbed curve by choosing a stretch factor that relates increments in arc length along the reference curve and the perturbed curve. This is incorporated in a theory that is accurate to first order in the ray position and to second order in the travel time. The second order travel time perturbation describes the effect of changes in the position of the ray on the travel time. For first arrivals, rays are paths of minimum travel time. The travel time along a ray estimate is therefore necessarily longer than the travel time along the true ray. The resulting travel time bias is described to leading order by the second order perturbation of the travel time. This quantity may be of great importance in nonlinear travel time tomography. Several existing perturbation equations are shown to correspond to the general equation derived in this paper for specific choices of the stretch factor. Depending on the stretch factor, analytical solutions for the ray perturbation can be found for different models of the reference slowness. For two-point ray tracing problems, the arc length (measured in terms of the independent parameter) may change with the ray perturbation. It is shown both numerically and analytically that for two-point ray tracing problems, one must introduce a tuning parameter in the perturbation problem. Leaving out such a tuning parameter, as is done in current Hamiltonian ray perturbation theory, may lead to erroneous solutions. In the formulation of this paper, paraxial ray perturbations, slowness perturbations, and pure ray bending are treated in a uniform fashion. This may be very useful in nonlinear tomographic inversions which include earthquake relocation.

1. INTRODUCTION

The accurate determination of travel times and ray paths is extremely important in seismology. Both in the determination of ray geometric Green's functions and in tomographic inversions, it is crucial to have accurate estimates of travel times and ray positions that can be determined efficiently. In many applications, rays (or ray estimates) are known in a slowness model, and perturbation theory is used to estimate the ray positions when the slowness or the initial (or boundary) values of the rays are perturbed.

The effect of perturbations of the initial conditions [*Cerveny et al.*, 1977, 1984; *Cerveny and Psencik*, 1979; *Chapman*, 1985] is called paraxial ray theory. This theory is used for the determination of amplitudes [e.g., *Cerveny et al.*, 1977] and for the determination of the Frechet derivatives of amplitudes with respect to slowness variations [*Neele et al.*, 1993a,b]. Perturbation theory has also been formulated to describe the effect of slowness perturbations on the ray position [*Dahlen and Henson*, 1985; *Farra and Madariaga*, 1987; *Moore*, 1991; *Virieux*, 1991; *Snieder and Sambridge*, 1992]. When the reference curve is not a true ray in the reference slowness field one speaks of true ray bending [*Julian and Gubbins*, 1977; *Pereyra et al.*, 1980; *Moser et al.*, 1992; *Farra*, 1992]. In all the studies cited here, an equation was derived for the first order perturbation of the ray position. An expression for the second order travel time perturbation was derived in *Snieder and Sambridge* [1992]. See *Sambridge and Snieder* [1993] for the accuracy of this expression for mantle tomography. The second order travel time perturbation may be very useful when applying ray perturbation theory to nonlinear travel time tomography.

In this paper, the theory of *Snieder and Sambridge* [1992], which employed ray-centered coordinates, is generalized to include a more general mapping from the reference curve to the perturbed curve. The notation in this paper is the same as in *Snieder and Sambridge* [1992]. Because of Fermat's theorem, the first order travel time perturbation does not depend on the perturbation of the ray position. Ray bending effects lead to a nonlinear dependence of the travel time on the perturbation of the slowness. This nonlinearity is described in this paper to

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leading order by the second order perturbation of the travel time. It is shown in section 6 that in order to compute the second order travel time perturbation it suffices to compute the first order perturbation in the ray position. For this reason, the theory is only presented for the first order perturbation in the ray position.

Why are there so many different formulations of ray perturbation theory? One reason is that the theories concerned with paraxial rays, slowness perturbations and pure ray bending address different problems. However, this is not the explanation, because *Snieder and Spencer* [1993] show how paraxial rays, slowness perturbations, and pure ray bending can be combined in a single theory that is accurate to first order in the ray position and to second order in the travel time.

The primary reason for the many different formulations of ray perturbation theory is that the perturbation problem for the ray position contains a fundamental ambiguity which arises because a point on the perturbed ray can be moved along the perturbed ray without altering the shape of the perturbed curve. This means that it is possible to devise different mappings between points on the reference and perturbed rays. The goal of this paper is to clarify this ambiguity and to develop tools to handle it in an explicit fashion. To this end, an equation for the first order ray perturbation is derived in section 3 which does not involve a specific mapping between reference and perturbed curves. The tools for specifying the mapping from the reference curve to the perturbed curve are developed in section 4. In this new formalism, one is able to prescribe the mapping from the reference curve to the perturbed curve explicitly. In previous work this mapping has been prescribed implicitly by the choice of the coordinate system or the independent parameter. The second order travel time perturbation is derived in section 6.

In the second half of the paper, it is shown how the equations of *Julian and Gubbins* [1977], *Moore* [1991], *Snieder and Sambridge* [1992] and *Farra* [1992] can be derived as special cases of the general ray perturbation equation by choosing different forms of the mapping from the reference curve to the perturbed curve. It is also shown that the existing equations for the ray perturbation derived from a Hamiltonian formalism lead to erroneous results when applied to two-point ray tracing problems.

2. Role of the Stretch Factor

Consider the perturbation problem where in a reference medium with slowness $u_0(\mathbf{r})$ one has an estimate $\mathbf{r}_0(s_0)$ for the ray position. The arc length s_0 along this unperturbed curve is used throughout this paper to parameterize the ray position. The arc length of the unperturbed curve is denoted by S_0. This quantity denotes the total arc length from source to receiver when the theory is applied to a whole ray, or it denotes the arc length of the reference curve through a cell when the theory is applied in a formalism based on a parameterization in cells [e.g., *Virieux*, 1991]. As in *Snieder and Spencer* [1993], one or a combination of the following perturbations can be considered:

1. The slowness may be perturbed

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \varepsilon u_1(\mathbf{r}) \ . \tag{1}$$

2. The curve in the reference medium may not be a true ray

$$\nabla u_0 - \frac{d}{ds_0}\left(u_0 \frac{d\mathbf{r}_0}{ds_0}\right) = \varepsilon \mathbf{R}_b \ . \tag{2}$$

The quantity $\mathbf{R}_b$ measures the degree to which the reference curve violates the equation of kinematic ray tracing in the reference medium. Since the starting curve need not be a true ray in the reference medium, the term "reference curve" is used rather than "reference ray".

3. The end-points of the curve may be perturbed

$$\mathbf{r}(0) = \mathbf{r}_0(0) + \varepsilon \mathbf{a} \qquad \mathbf{r}(S_0) = \mathbf{r}_0(S_0) + \varepsilon \mathbf{b} \ . \tag{3}$$

The parameter ε is used to denote that these effects are assumed to be small, and facilitates a systematic perturbation approach. The aim of ray perturbation theory is to investigate the effect of the perturbations (1)-(3) on the ray position and travel time. Let the ray position under these perturbations be given by

$$\mathbf{r}(s_0) = \mathbf{r}_0(s_0) + \varepsilon \mathbf{r}_1(s_0) + \varepsilon^2 \mathbf{r}_2(s_0) + \ldots\ldots \tag{4}$$

The perturbed ray is parameterized with the arc length s_0 along the unperturbed ray.

It is crucial to note at this point that one can move the points along a curve without altering the shape of the curve. This means that there exists an ambiguity in the mapping of the points along the reference curve onto the perturbed curve because one may associate a point on the reference curve with different points along the perturbed curve and obtain the same perturbed curve. As long as one ensures that the boundary conditions (3) are satisfied, infinitely many mappings can be defined from the reference curve onto the perturbed curve. In other words, the perturbation (4) is not defined uniquely without a specific mapping of the reference curve to the perturbed curve. In *Snieder and Sambridge* [1992] this mapping is specified by the ray-centered coordinates they employed, i.e., the perturbation was constrained to be perpendicular to the reference curve (see Figure 1*a*). In contrast to this, *Moore* [1991] defined the mapping by the requirement that a point at arc length s_0 along the reference curve was mapped onto a point with the same arc length along the perturbed curve (see Figure 1*b*). *Snieder and Spencer* [1993] used the relative arc length to define the mapping from unperturbed to perturbed curve as shown in Figure 1*c*. (The relative arc length is defined as the acrlength measured along the curve to the point under consideration divided by the total arc length.) These different mappings necessarily lead to different perturbation equations because the perturbation $\mathbf{r}_1$ depends on the employed mapping.

In general, the increments ds_0 and ds along the unperturbed and the perturbed curve are different. This is important for the equation of kinematic ray tracing

$$\frac{d}{ds}\left(u \frac{d\mathbf{r}}{ds}\right) = \nabla u \ , \tag{5}$$

because the acrlength ds appears in the derivatives. This is also the case for the travel time

$$T = \int u \, ds \ , \tag{6}$$

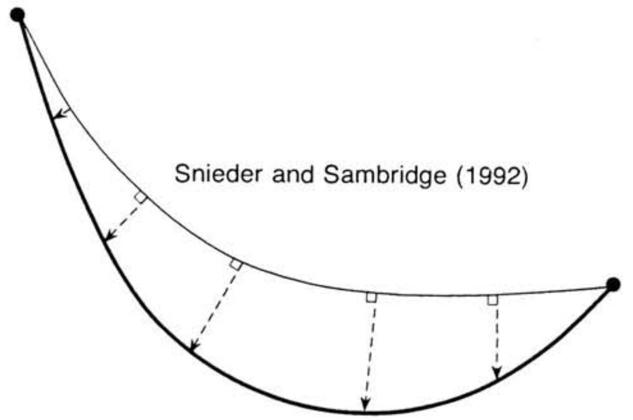

Fig. 1*a*. Mapping used by *Snieder and Sambridge* [1992] where each point on the reference curve is perturbed perpendicular to the reference curve. The reference curve is shown by a thin solid line, the perturbed curve with a thick solid line. The mapping is indicated by the dashed arrows.

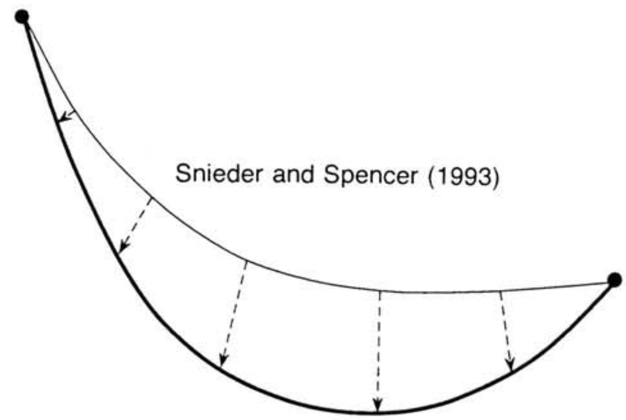

Fig. 1*c*. Mapping used by *Snieder and Spencer* [1993] where the mapping is defined by the requirement that corresponding points on the reference curve and the perturbed curve have equal relative arc length to the source. Line styles are as defined in Figure 1*a*.

because of the presence of the increment ds.

A dot is used throughout this paper to denote differentiation with respect to the unperturbed arc length s_0. The vector $\dot{\mathbf{r}}_0$ is of unit length because $\dot{\mathbf{r}}_0 \equiv d\mathbf{r}_0 / ds_0$. From the relation $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_0) = 1$ it follows by differentiation that

$$(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_0) = 0 \ . \tag{7}$$

The derivative of a quantity $\xi(\mathbf{r})$ along the reference curve is given by $\dot{\xi} \equiv d\xi / ds_0 = (\dot{\mathbf{r}}_0 \cdot \nabla \xi)$.

It is important to note that in general the perturbed curve has a different arc length than the unperturbed curve. The increment in arc length ds along the perturbed curve is equal to the length of the vector $d\mathbf{r}$, so that $(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}$. Dividing this by $(ds_0)^2$ leads to the relation $(\partial s/\partial s_0)^2 = [\, d\mathbf{r}/ds_0 \cdot d\mathbf{r}/ds_0 \,]$. Inserting the perturbation series (4) and using the fact that $\dot{\mathbf{r}}_0$ has unit length, one finds that to second order

$$\frac{\partial s}{\partial s_0} = 1 + \varepsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) + \varepsilon^2\left(\frac{1}{2}(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) - \frac{1}{2}(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)^2 + (\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_2)\right) , \tag{8}$$

and its reciprocal expression

$$\frac{\partial s_0}{\partial s} = 1 - \varepsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) - \varepsilon^2\left(\frac{1}{2}(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) - \frac{3}{2}(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)^2 + (\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_2)\right) . \tag{9}$$

To first order, the unperturbed and the perturbed arc length are related by

$$\frac{\partial s}{\partial s_0} = 1 + \varepsilon(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0) \ . \tag{10}$$

We call the product $(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0)$ the stretch factor, since this term describes how an increment ds_0 along the unperturbed curve is related to an increment ds along the perturbed curve. We shall see that this factor can be used to specify the mapping from the unperturbed to the perturbed curve. Integrating (10) with repsect to s_0 one obtains to first order

$$s(s_0) = s_0 + \varepsilon \int_0^{s_0} (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0)\, ds_0' \ . \tag{11}$$

Hence the stretch factor unambiguously defines the relation between the arc length of corresponding points on the unperturbed and the perturbed curves.

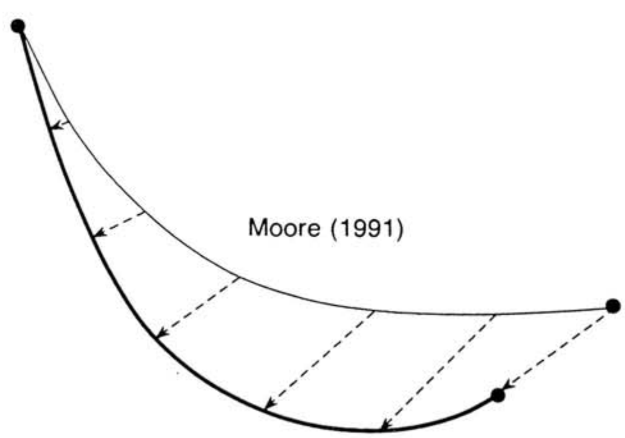

Fig. 1*b*. Mapping used by *Moore* [1991] where the mapping is defined by the requirement that corresponding points on the reference curve and the perturbed curve have equal arc length to the source. Note that in general the perturbed ray will not end at the receiver. Line styles are as defined in Figure 1*a*.

3. Perturbation Equation

The starting point for the derivation of the ray perturbation is the equation of kinematic ray tracing (5). Using the ray perturbation (4), one obtains with a Taylor expansion of $u_0(\mathbf{r})$ and $u_1(\mathbf{r})$ around $\mathbf{r}_0$ that the slowness (1) is given to second order by

$$u(\mathbf{r}) = u_0(\mathbf{r}_0) + \varepsilon\,(\, u_1(\mathbf{r}_0) + \mathbf{r}_1 \cdot \nabla u_0(\mathbf{r}_0)\,)$$
$$+ \varepsilon^2\left(\mathbf{r}_1 \cdot \nabla u_1(\mathbf{r}_0) + \frac{1}{2}\mathbf{r}_1\mathbf{r}_1 : \nabla\nabla u_0(\mathbf{r}_0) + \mathbf{r}_2 \cdot \nabla u_0(\mathbf{r}_0)\right) , \tag{12}$$

where : stands for a double contraction. The slowness gradient is given to first order by

$$\nabla u(\mathbf{r}) = \nabla u_0(\mathbf{r}_0) + \varepsilon\,(\,\nabla u_1(\mathbf{r}_0) + \mathbf{r}_1 \cdot \nabla\nabla u_0(\mathbf{r}_0)\,) \ . \tag{13}$$

The first order expansion of both the slowness and its gradient are needed to determine the first order ray perturbation from equation (5). The second order expansion of the slowness is needed to derive the second order travel time perturbation from equation (6).

The perturbation expansions (4), (12) and (13) can be inserted in the equation of kinematic ray tracing (5). Taking into account that the arc lengths ds and ds_0 can be different, one can derive the following differential equation for the first order ray perturbation:

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) - u_0\dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1) + \left(\dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\nabla u_0) - 2\nabla u_0\right)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1)$$

$$+\left(\ddot{\mathbf{r}}_0\nabla u_0 + \dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\nabla\nabla u_0) - \nabla\nabla u_0\right)\cdot\mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1 \ . \quad (14)$$

This expression constitutes the $O(\varepsilon)$ contribution to the perturbation expansion of equation (5) of kinematic ray tracing. The derivation of this expression is shown in Appendix A. The quantity $\mathbf{R}_1$ is defined by

$$\mathbf{R}_1 \equiv \nabla u_1 - \frac{d}{ds_0}\left(u_1\frac{d\mathbf{r}_0}{ds_0}\right) . \quad (15)$$

By carrying out the differentiation on the right-hand side, $\mathbf{R}_1$ becomes

$$\mathbf{R}_1 = \nabla u_1 - \dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\nabla u_1) - u_1\ddot{\mathbf{r}}_0 \ . \quad (16)$$

This expression can be simplified by using the fact that to order ε the curve $\mathbf{r}_0(s_0)$ satisfies the equation of kinematic ray tracing in the reference medium, i.e.,

$$\nabla u_0 - u_0\ddot{\mathbf{r}}_0 - \frac{du_0}{ds_0}\dot{\mathbf{r}}_0 = O(\varepsilon) \ , \quad (17)$$

which follows from (2). Using this expression to eliminate $\ddot{\mathbf{r}}_0$ in (16), one finds that

$$\mathbf{R}_1 = \nabla u_1 - \frac{u_1}{u_0}\nabla u_0 - \dot{\mathbf{r}}_0\dot{\mathbf{r}}_0\cdot(\nabla u_1 - \frac{u_1}{u_0}\nabla u_0) + O(\varepsilon) \ . \quad (18)$$

With the definition $\nabla_T \equiv \nabla - \dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\nabla)$ for the derivative perpendicular to the reference curve, one obtains to leading order that

$$\mathbf{R}_1 = u_0\nabla_T\left(\frac{u_1}{u_0}\right) . \quad (19)$$

Moreover, one can show that: $\varepsilon(\dot{\mathbf{r}}_0\cdot\mathbf{R}_b) = \dot{\mathbf{r}}_0\cdot(\nabla u_0 - u_0\ddot{\mathbf{r}}_0 - \dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\nabla u_0)) = 0$, by virtue of the fact that $\dot{\mathbf{r}}_0$ is of unit length and of (7). This implies that to leading order the vectors $\mathbf{R}_b$ and $\mathbf{R}_1$ are orthogonal to the reference curve:

$$(\dot{\mathbf{r}}_0\cdot\mathbf{R}_b) = (\dot{\mathbf{r}}_0\cdot\mathbf{R}_1) = 0 \ . \quad (20)$$

As shown by *Snieder and Sambridge* [1992], there is a close analogy between equation (14) for the ray perturbation and the equation of motion in classical mechanics for a point mass: $d(m\dot{\mathbf{r}})/dt = \mathbf{F}$. If we identify s_0 with time, and u_0 with mass, then $u_0\dot{\mathbf{r}}_1$ corresponds to momentum. The first term $d(u_0\dot{\mathbf{r}}_1)/ds_0$ in (14) then corresponds to the derivative of the momentum. In this analogy the contributions $\mathbf{R}_b$ and $\mathbf{R}_1$ on the right-hand side act as external forces. The terms containing $\mathbf{r}_1$ are analogous to linear elastic restoring forces, since they correspond to forces that are linear in the displacement $\mathbf{r}_1$. Similarly, the terms containing $\dot{\mathbf{r}}_1$ in the left-hand side of equation (14) correspond in this analogy to forces that are linear in the velocity, they can therefore be regarded as generalized frictional forces.

At this point, the mapping from the unperturbed curve onto the perturbed curve has not been specified. That an ambiguity exists can be seen by rewriting the first two terms of (14) in the form

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) - u_0\dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1) = u_0(\ddot{\mathbf{r}}_1 - \dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1)) + \frac{du_0}{ds_0}\dot{\mathbf{r}}_1 \quad (21)$$

Using the classical mechanics terminology, the term $du_0/ds_0\,\dot{\mathbf{r}}_1$ is an inertia term which accounts for the fact that the "mass" u_0 of the point mass is not conserved. The terms $\ddot{\mathbf{r}}_1 - \dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1)$ in the right-hand side of (21) denote the component of the acceleration $\ddot{\mathbf{r}}_1$ perpendicular to the reference curve because it is the difference of the acceleration $\ddot{\mathbf{r}}_1$ with the projection of the acceleration on the reference curve $\dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1)$. This means that only the component of the acceleration perpendicular to the reference curve is determined by equation (14). Note that because of expression (20), the external force $\mathbf{R}_b + \mathbf{R}_1$ has a vanishing component along the reference curve. The component of the acceleration along the reference curve is not determined by equation (14), and no external forces are acting in the direction of the reference curve. This means that the motion along the reference curve is not determined by equation (14). This is a manifestation of the ambiguity in ray perturbation theory that is due to the fact that one can move the points along a curve without changing the shape of the curve. In other words, equation (14) does not completely specify the mapping from the unperturbed curve to the perturbed curve because only the motion of the perturbation $\mathbf{r}_1$ perpendicular to the reference curve is determined by equation (14). The specification of the mapping from the unperturbed curve is discussed further in section 4.

It is interesting to note that the original (nonlinear) equation (5) also leaves the position of the points $\mathbf{r}(s)$ along the ray unspecified. Dotting (5) with the vector $d\mathbf{r}/ds$ and using the fact that this unit vector is orthogonal to $d^2\mathbf{r}/ds^2$, one finds that the component of (5) along the ray reduces to the identity

$$\frac{du}{ds} = \frac{d\mathbf{r}}{ds}\cdot\nabla u \ . \quad (22)$$

This equation is simply the definition of the directional derivative d/ds of a function which depends on $\mathbf{r}$ only. The important point to note is that this expression does not contain any information of the ray position, since (22) holds for any vector $\mathbf{r}(s)$. This means that the location of points along the ray is determined by the condition $|d\mathbf{r}/ds| = 1$ rather than the equation of kinematic ray tracing.

4. CHOOSING THE STRETCH FACTOR

Up to this point, the perturbation equations do not provide an unambiguous description of the perturbation of the reference curve. The reason for this is that the stretch factor which controls the mapping from the unperturbed to the perturbed curve has not yet been specified. The analogy between the perturbation equation and the motion of a point mass in classical mechanics may clarify this issue.

As shown in section 3, the component of the acceleration along the reference curve is not determined by equation (14). In other words, since the $\ddot{\mathbf{r}}_1$ terms in equation

(14) are projected onto the plane perpendicular to the reference curve, equation (14) represents a second order differential equation for the components of $\mathbf{r}_1$ perpendicular to the reference curve, but only a first order differential equation for the components of $\mathbf{r}_1$ along the reference curve. This means that equation (14) needs to be supplemented with five boundary conditions. This is inconsistent with the boundary conditions for two-point ray tracing problems, which impose six constraints on the ray perturbation $\mathbf{r}_1$. (The coordinates of the two end-points.) Note also that because of (20) the component of the external forces along the reference curve $\dot{\mathbf{r}}_0 \cdot (\mathbf{R}_b + \mathbf{R}_1)$ vanishes. Therefore it is necessary to prescribe the motion along the reference curve by defining the stretch factor $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)$.

It is assumed here that the stretch factor depends on $\mathbf{r}_1$ but not on $\dot{\mathbf{r}}_1$ or $\ddot{\mathbf{r}}_1$. The stretch factor may also contain a term independent of the ray perturbation. To maintain a first order theory for the ray perturbation, the stretch factor must depend linearly on $\mathbf{r}_1$. A general form which satsfies these conditions is

$$u_0(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) = (\mathbf{g} \cdot \mathbf{r}_1) + F \ , \tag{23}$$

The vector $\mathbf{g}$ and the scalar F may be functions of s_0, and the factor u_0 is introduced for notational convenience. The stretch factor (23) cannot be used in this form to remove the parallel acceleration $(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_1)$ from (14). However, by differentiation of (23) and using equation (17), which states that to leading order $d(u_0\dot{\mathbf{r}}_0)\,/\,ds_0 = \nabla u_0$, one obtains

$$u_0(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_1) = (\dot{\mathbf{g}} \cdot \mathbf{r}_1) + (\mathbf{g} - \nabla u_0) \cdot \dot{\mathbf{r}}_1 + \dot{F} \ . \tag{24}$$

Inserting (23) and (24) in (14) and using the fact that $\dot{\mathbf{g}} = \dot{\mathbf{r}}_0 \cdot \nabla \mathbf{g}$, one obtains the following equation for the ray perturbation:

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) - \dot{\mathbf{r}}_0\,(\mathbf{g} - \nabla u_0) \cdot \dot{\mathbf{r}}_1 + \Bigg(\ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0\,\dot{\mathbf{r}}_0 \cdot \nabla(\nabla u_0 - \mathbf{g}) - \nabla\nabla u_0 + \frac{1}{u_0}(\dot{\mathbf{r}}_0 \cdot \nabla u_0)\dot{\mathbf{r}}_0\,\mathbf{g} - \frac{2}{u_0}\nabla u_0 \mathbf{g} \Bigg) \cdot \mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1 + u_0 \frac{d}{ds_0}\left(\frac{F}{u_0}\right)\dot{\mathbf{r}}_0 + \frac{2F}{u_0}\nabla u_0 \ . \tag{25}$$

Note that this expression has additional forcing terms in the right-hand side compared to the original equation (14). These forcing terms depend on F, while the frictional and linear restoring force depend on the choice of $\mathbf{g}$. The reason for this can be seen by using the classical mechanical interpretation, which states that the stretch factor (23) effectively prescribes the component of the momentum $u_0\dot{\mathbf{r}}_1$ of the point mass along the reference curve. To realize this prescribed momentum, additional forces are needed.

In equation (25), the choice of $\mathbf{g}$ and F is not yet specified. The perturbation equations of *Julian and Gubbins* [1977], *Chapman* [1985], *Moore* [1991], *Virieux* [1991], *Farra* [1992], and *Snieder and Spencer* [1993] can be derived from (25) using special choices of $\mathbf{g}$ and F. The correspondence of equation (25) with existing schemes for ray perturbation theory and ray bending is shown explicitly in sections 7-10.

Equation (25) was derived by inserting the stretch factor (23) in expression (14). For reasons of consistency it should be checked whether the solution of (25) satisfies the expression (23) for the stretch factor. One can show that if the solution of (25) satisfies (23) at one point along the curve, then (23) is satisfied everywhere along the curve. To see this, define

$$K(s_0) \equiv u_0(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) - (\mathbf{g} \cdot \mathbf{r}_1) - F \ . \tag{26}$$

The condition $K = 0$ implies that (23) is satisfied. By direct differentiation of (26), one finds using (25) to eliminate the term, $d(u_0\dot{\mathbf{r}}_1)\,/\,ds_0$ that

$$\frac{dK}{ds_0} = \left(u_0\ddot{\mathbf{r}}_0 - \nabla u_0 \right) \cdot \dot{\mathbf{r}}_1 + \frac{1}{u_0}\frac{du_0}{ds_0}(\mathbf{g} \cdot \mathbf{r}_1 + F) \ , \tag{27}$$

where (7), (20) and the fact that $\dot{\mathbf{r}}_0$ is a unit vector have been used. Using the definition (26), this can be written as

$$\frac{dK}{ds_0} = -\frac{1}{u_0}\frac{du_0}{ds_0} K + \left(\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_0) - \nabla u_0 \right) \cdot \dot{\mathbf{r}}_1 \ . \tag{28}$$

By virtue of (2) the last term is of order ε and can thus be ignored. This means that to leading order

$$\frac{d}{ds_0}(u_0 K) = 0 \ . \tag{29}$$

When (23) is satisfied at one point along the reference curve, e.g., $K = 0$ at the source, then direct integration of (29) shows that to first order K is identical to zero at all points along the curve. Therefore the solution of (25) guarantees that if the stretch factor is given by (23) at a single point along the curve, then the same expression holds along the entire curve.

One caveat should be made here. Equation (29) is valid to leading order in ε. In iterative applications as shown by *Snieder and Spencer* [1993], the $O(\varepsilon)$ terms that are ignored might accumulate. In such a situation it may be prudent to resample the points along the curve between iterations in order to prevent this cumulative source of error. However, in the numerical examples of *Snieder and Spencer* [1993], a resampling was not found to be necessary.

It is important to note that for initial value ray tracing any value of the quantities $\mathbf{g}$ and F can be used without violating the boundary conditions. This is because equation (23) only determines the component of $\dot{\mathbf{r}}_1$ along the reference curve and so the other five boundary conditions (i.e. the initial perturbation, $\mathbf{r}_1(0)$ and two take-off angles, or equivalently, the components of $\dot{\mathbf{r}}_1$ perpendicular to the reference curve), may be specified independently. Nevertheless one should realize that for a given choice of $\mathbf{g}$ and F the perturbed ray will, in general, not hit a given receiver. This case is illustrated in Figure 1b for the stretch factor $\mathbf{g} = 0$, $F = 0$, which, as it will be seen in section 7, corresponds to the mapping used by *Moore* [1991].

For two-point ray tracing, both end-points $\mathbf{r}_1(0)$ and $\mathbf{r}_1(S_0)$ are specified, which makes a total of six boundary conditions. Since the constraint (23) must also hold at the end-points, an extra boundary condition on the quantity $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)$ is introduced by (23), and an arbitrary choice of $\mathbf{g}$ and F will result in an overdetermined system. To overcome this problem, a degree of freedom must be included in the prescription of $\mathbf{g}$ and F.

5. Numerical Solution of the Perturbation Equation

Equation (25) for the ray perturbation lends itself well for obtaining analytical solutions for special choices of the reference slowness u_0 and the terms g and F in the stretch factor (23). However, as argued in the previous section, for two-point ray tracing problems one must leave one degree of freedom in the specification of the stretch factor. This implies that equation (25) must contain one unspecified constant. For applications where an analytical solution of (25) can be derived it is no problem to leave an unspecified constant in the equation for the ray perturbation. As an illustration, consider the choice

$$\mathbf{g} = \nabla u_0 \qquad F = u_0 C \ , \tag{30}$$

where C is a constant. Inserting this in (25) and using (17) to eliminate $\ddot{\mathbf{r}}_0$ to leading order in ε, one obtains with the identity $\nabla\nabla u_0 + \nabla u_0 \nabla u_0 = 1/(2u_0)\, \nabla\nabla u_0^2$ that

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) - \frac{1}{2u_0}\nabla\nabla u_0^2 \cdot \mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1 + 2C\nabla u_0 \ . \tag{31}$$

For a medium where the squared slowness has a linear gradient

$$u_0^2 = A + \Gamma \cdot \mathbf{r} \ , \tag{32}$$

equation (31) reduces to

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) = \mathbf{R}_b + \mathbf{R}_1 + 2C\nabla u_0 \ . \tag{33}$$

This equation can be solved in closed form regardless of the value of C.

However, when solving equation (25) for the ray perturbation numerically, it is undesirable when the equation contains an unspecified constant. Fortunately, this complication can be avoided. When deriving (25), both terms $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)$ and $(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_1)$ were eliminated from (14) by using the stretch factor (23) and its derivative (24) respectively. Suppose for the moment that one only eliminates the term $(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_1)$ from (14) using (24) and retains the $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)$ terms; this gives

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) + \left(\dot{\mathbf{r}}_0 (\nabla u_0 - \mathbf{g}) - 2\nabla u_0 \dot{\mathbf{r}}_0 \right) \cdot \dot{\mathbf{r}}_1$$
$$+ \left(\ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0 \dot{\mathbf{r}}_0 \cdot \nabla(\nabla u_0 - \mathbf{g}) - \nabla\nabla u_0 \right) \cdot \mathbf{r}_1$$
$$= \mathbf{R}_b + \mathbf{R}_1 - \left(\frac{du_0}{ds_0}(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) - \dot{F} \right) \dot{\mathbf{r}}_0 \ . \tag{34}$$

The simplest way to incorporate the required degree of freedom in the stretch factor is to use F as a tuning parameter. Note that F only appears in the last term of (34). For numerical applications it is preferable to use an equation where this tuning parameter does not appear explicitly. Two different forms of F can be used to achieve this.

Option 1: Using (23), the last term on the right-hand side of (34) can be written as

$$\frac{du_0}{ds_0}(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) - \dot{F} = \frac{1}{u_0}\frac{du_0}{ds_0}(\mathbf{g} \cdot \mathbf{r}_1) - u_0 \frac{d}{ds_0}\left(\frac{F}{u_0}\right) \ . \tag{35}$$

With the choice $F = u_0 C$, with C an unspecified constant, the last term in (35) vanishes, and so (34) becomes

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) + \left(\dot{\mathbf{r}}_0 (\nabla u_0 - \mathbf{g}) - 2\nabla u_0 \dot{\mathbf{r}}_0 \right) \cdot \dot{\mathbf{r}}_1$$
$$+ \left(\ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0 \dot{\mathbf{r}}_0 \cdot \nabla(\nabla u_0 - \mathbf{g}) - \nabla\nabla u_0 + \frac{1}{u_0}\frac{du_0}{ds_0}\dot{\mathbf{r}}_0 \mathbf{g} \right) \cdot \mathbf{r}_1$$
$$= \mathbf{R}_b + \mathbf{R}_1 \qquad F = u_0 C \ . \tag{36}$$

In this equation, the required degree of freedom, which is contained in the constant C, does not appear explicitly in the equation (36) for the ray perturbation.

Option 2: An alternative way to remove the F from equation (34) for the ray update is to make it a constant. Since only the derivative $\dot{F}$ appears in (34) this gives

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) + \left(\dot{\mathbf{r}}_0 (\nabla u_0 - \mathbf{g}) + \frac{du_0}{ds_0}\dot{\mathbf{r}}_0 \dot{\mathbf{r}}_0 - 2\nabla u_0 \dot{\mathbf{r}}_0 \right) \cdot \dot{\mathbf{r}}_1$$
$$+ \left(\ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0 \dot{\mathbf{r}}_0 \cdot \nabla(\nabla u_0 - \mathbf{g}) - \nabla\nabla u_0 \right) \cdot \mathbf{r}_1$$
$$= \mathbf{R}_b + \mathbf{R}_1 \qquad F = const \ . \tag{37}$$

In this expression, F is the constant that can be tuned to the boundary conditions.

Note that at this point one is still completely free in the choice of g. The price one pays for eliminating F from the perturbation equation is that (36) and (37) contain the product $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)$ explicitly. However, the presence of the $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)$ term poses no additional problems when solving one of the equations (36) or (37) numerically.

F has been removed from both equations above, but can be evaluated by inserting the numerical solution of (36) or (37) in equation (23) for the stretch factor. Because of (29) one only needs to do this for a single point along the ray. However, it should be noted that when using (36) or (37), there is no need to compute F explicitly.

6. Second Order Travel Time Perturbation

In this section, an expression for the the second order perturbation of the travel time is derived. The derivation uses the general expression (14), hence the final result can be used for any choice of the stretch factor. Under the three types of perturbations (1-3), the travel time can be expressed as a perturbation series:

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots\ldots \ . \tag{38}$$

The second order travel time is of interest because it handles the effect of ray bending on the travel time, and hence the travel time bias [*Snieder and Sambridge*, 1992; *Nolet and Moser*, 1993; *Roth et al.*, 1993]. As shown in equation (13) of *Snieder and Sambridge* [1992], the travel time is given by

$$T = \int_0^{S_0} u_0(\mathbf{r}_0)\, ds_0 + \varepsilon \int_0^{S_0} \left(u_1(\mathbf{r}_0) + (\mathbf{r}_1 \cdot \nabla u_0) + (\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) u_0 \right) ds_0$$
$$+ \varepsilon^2 \int_0^{S_0} \left(\frac{u_0}{2}(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) - \frac{u_0}{2}(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0)^2 + (u_1 + (\mathbf{r}_1 \cdot \nabla u_0))(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) \right.$$
$$\left. + (\mathbf{r}_1 \cdot \nabla u_1) + \frac{1}{2}(\mathbf{r}_1 \mathbf{r}_1 : \nabla\nabla u_0) + (\mathbf{r}_2 \cdot \nabla u_0) + (\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_2) u_0 \right) ds_0 + \cdots \tag{39}$$

The last two terms of the order ε and order ε^2 contributions cancel to leading order in ε. To see this, consider the integral

$$I \equiv \int_0^{S_0} \left(\xi \cdot \nabla u_0) + (\dot{\mathbf{r}}_0 \cdot \dot{\xi}) u_0 \right) ds_0 \ , \tag{40}$$

where $\xi(s_0)$ is an arbitrary vector function along the reference curve. The term ∇u_0 can be eliminated using (2), so that

$$I = \int_0^{S_0} \left(\xi \cdot \frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_0) + (\dot{\mathbf{r}}_0 \cdot \dot{\xi}) u_0 \right) ds_0 + \varepsilon \int_0^{S_0} \xi \cdot \mathbf{R}_b \, ds_0 \tag{41}$$

Applying an integration by parts to the first term gives

$$\int_0^{S_0} \left((\xi \cdot \nabla u_0) + (\dot{\mathbf{r}}_0 \cdot \dot{\xi}) u_0 \right) ds_0 = \left[u_0 (\dot{\mathbf{r}}_0 \cdot \xi) \right]_0^{S_0} + \varepsilon \int_0^{S_0} (\xi \cdot \mathbf{R}_b) \, ds_0 \tag{42}$$

where

$$\left[f \right]_0^{S_0} \equiv f(S_0) - f(0) \ . \tag{43}$$

Expression (42) can be used to eliminate the first two terms in the order ε contribution in (39) by inserting $\xi = \mathbf{r}_1$. This gives a boundary term of order ε and a term containing $\mathbf{R}_b$ that spills over to the second order travel time perturbation. Similarly, the $\mathbf{r}_2$ terms vanish from the order ε^2 term in (39) when (42) is used with $\xi = \mathbf{r}_2$. In this case the boundary terms do not contribute because the boundary perturbation are forced to be absorbed by the first order perturbation; see (3). This implies that in order to compute the travel time to second order one only needs to consider the ray perturbation to first order. Using these results the first and second order travel time perturbations are given by

$$T_1 = \int_0^{S_0} u_1 \, ds_0 + [u_0 (\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)]_0^{S_0} \ , \tag{44a}$$

$$T_2 = \int_0^{S_0} \left(\frac{u_0}{2} (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) - \frac{u_0}{2} (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0)^2 + (u_1 + \mathbf{r}_1 \cdot \nabla u_0)(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) \right.$$

$$\left. + \mathbf{r}_1 \cdot \nabla u_1 + \frac{1}{2} (\mathbf{r}_1 \mathbf{r}_1 : \nabla\nabla u_0) + (\mathbf{r}_1 \cdot \mathbf{R}_b) \right) ds_0 \ . \tag{44b}$$

The first order travel time perturbation T_1 contains the integral of the slowness perturbation along the reference ray. This term forms the basis of linearized travel time tomography. The term $[u_0(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)]_0^{S_0}$ describes the travel time change due to the extension of the ray at the endpoints in the direction of the reference curve. The second order travel time perturbation T_2 contains second order terms in $\mathbf{r}_1$ and cross terms containing u_1 and $\mathbf{r}_1$. In addition to this, it also contains a term $(\mathbf{r}_1 \cdot \mathbf{R}_b)$, which accounts for the fact that the reference ray need not be a true ray. For this reason, equation (42) can be regarded as a generalization of Fermat's theorem, where the fact that the reference curve need not be a true ray is explicitly accounted for by the term $\varepsilon \mathbf{R}_b$.

It is shown in Appendix B that the integral in (44b) can be reduced to a simpler integral containing $\mathbf{r}_1$ and $\mathbf{R}_b + \mathbf{R}_1$ only:

$$T_2 = \frac{1}{2} \int_0^{S_0} \mathbf{r}_1 \cdot (\mathbf{R}_b + \mathbf{R}_1) \, ds_0 + \frac{1}{2} \Big[u_0 (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1) + (\mathbf{r}_1 \cdot \nabla u_0)(\mathbf{r}_1 \cdot \dot{\mathbf{r}}_0)$$

$$- u_0 (\mathbf{r}_1 \cdot \dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0) + 2 u_1 (\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1) \Big]_0^{S_0} \ . \tag{45}$$

This means that the second order travel time perturbation can be computed with a single integration along the reference curve once $\mathbf{r}_1$ is computed. Alternatively, one can eliminate the forcing term $(\mathbf{R}_b + \mathbf{R}_1)$ from (45) using (B8) of Appendix B, this gives an expression for the second order travel time perturbation that is explicitly quadratic in $\mathbf{r}_1$ and $\dot{\mathbf{r}}_1$:

$$T_2 = \frac{1}{2} \int_0^{S_0} \Big(- u_0 (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) + u_0 (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0)^2$$

$$- 2 (\mathbf{r}_1 \cdot \nabla u_0)(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) - (\mathbf{r}_1 \mathbf{r}_1 : \nabla\nabla u_0) \Big) ds_0 \tag{46}$$

$$+ \Big[u_0 (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1) + (\mathbf{r}_1 \cdot \nabla u_0)(\mathbf{r}_1 \cdot \dot{\mathbf{r}}_0) - u_0 (\mathbf{r}_1 \cdot \dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_0) + u_1 (\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1) \Big]_0^{S_0}$$

When deriving the expressions (45) or (46) for the second order travel time perturbation, only the general equation (14) for the ray perturbation has been used. This means that the expressions for the second order travel time perturbation are valid for any choice of $\mathbf{g}$ and F. When the ray perturbation $\mathbf{r}_1$ is computed (either analytically or numerically) for a specific choice for $\mathbf{g}$ and F, one can determine the travel time bias by inserting the solution $\mathbf{r}_1$ in (45) or (46).

7. Relation With the Work of Moore [1991]

In the perturbation theory of *Moore* [1991], the distinction between the arc length along the reference curve and the perturbed curve is not made. This prompted *Snieder and Sambridge* [1992] to state that the theory of *Moore* [1991] is inconsistent. However, there are no consistency problems with her theory when one realizes that points along the reference curve are mapped onto points along the perturbed curve with equal arc length:

$$s(s_0) = s_0 \ . \tag{47}$$

From (10) and (23), one finds that the mapping of *Moore* [1991] corresponds to

$$\mathbf{g} = 0 \qquad F = 0 \ . \tag{48}$$

In order to see the relation of the perturbation equation of *Moore* [1991] and this work, one should make the following identification between her notation and the notation of this paper: $\mathbf{x}^\varepsilon \to \mathbf{r}_1$, $\mathbf{x}_0 \to \mathbf{r}_0$, $d/ds \to d/ds_0$. Furthermore, using her equation (11) and equation (19) of this paper gives $u_0 \nabla^T f \to u_0 \nabla_T (u_1 / u_0) = \mathbf{R}_1$. This means that equation (18) of *Moore* [1991] is in the notation of this paper given by

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) + \frac{d}{ds_0}(\dot{\mathbf{r}}_0 \nabla u_0 \cdot \mathbf{r}_1) - \nabla\nabla u_0 \cdot \mathbf{r}_1 = \mathbf{R}_1 \ . \tag{49}$$

Carrying out the differentiation of the second term gives

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) + \dot{\mathbf{r}}_0 \nabla u_0 \cdot \dot{\mathbf{r}}_1 + \left(\ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0 (\dot{\mathbf{r}}_0 \cdot \nabla\nabla u_0) - \nabla\nabla u_0 \right) \cdot \mathbf{r}_1 = \mathbf{R}_1 \ . \quad (50)$$

This expression can be obtained from (25) by inserting $\mathbf{R}_b = 0$, $\mathbf{g} = 0$ and $F = 0$. Therefore the theory of *Moore* [1991] follows as a special case of (25) using (48) for the stretch factor. The term $\mathbf{R}_b$ is not present in the theory of *Moore* [1991] because she assumed that the starting curve is a ray in the reference medium, and in that case $\mathbf{R}_b = 0$.

Note that in the theory of *Moore* [1991] the stretch factor is completely fixed. For two-point ray tracing problems the absence of a tuning parameter in the stretch factor corresponds with the fact that the end-point of the reference curve (the receiver) will not be mapped onto itself (see Figure 1*b*). Physically, this happens because in general the unperturbed ray and the perturbed ray have different arc lengths. This can be remedied by using (36) or (37) with the choice $\mathbf{g} = 0$, but with C or F constant.

In the theory of *Dahlen and Henson* [1985], which employs ray tracing of surface waves on an aspherical Earth, the difference in arc length between the reference curve and the perturbed curve is also not accounted for. The criterion (58) of *Dahlen and Henson* [1985] for closed orbits should therefore not be evaluated for $\Delta = 2\pi$, but for the arc length of the perturbed ray (measured in radians). However, since the arc length of the employed reference curve (a great circle) is stationary, this effect is of second order and can be ignored in their application to the computation of asymptotic eigenfrequencies of an aspherical Earth.

8. Relation With the Theory of Julian and Gubbins [1977]

In the perturbation theory of *Julian and Gubbins* [1977], points with equal relative arc length along the original curve and the perturbed curve are mapped onto each other. This implies that in their work

$$\frac{\partial s}{\partial s_0} = const \ . \quad (51)$$

With (10) and (23) this means that

$$\mathbf{g} = 0 \qquad F = u_0 C \ , \quad (52)$$

with C constant. Using these values for $\mathbf{g}$ and F in (36) gives

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) + \left(\dot{\mathbf{r}}_0 \nabla u_0 - 2\nabla u_0 \dot{\mathbf{r}}_0 \right) \cdot \dot{\mathbf{r}}_1 + \left(\ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0 (\dot{\mathbf{r}}_0 \cdot \nabla\nabla u_0) - \nabla\nabla u_0 \right) \cdot \mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1 \ . \quad (53)$$

This equation is a generalization of the perturbation equation (50) for the stretch factor of *Moore* [1991], since the relative arc length is preserved in the mapping from the reference curve to the perturbed curve, but the scale factor between the arc length of the two curves is a tunable parameter. Physically this corresponds to the fact that we allow the reference curve and the perturbed curves to have different arc lengths.

Equation (53) is identical to the perturbation equation (17) of *Snieder and Spencer* [1993] derived for the stretch factor (51). They show that for the pure bending case ($\mathbf{R}_1 = 0$), the x and y components of (53) reduce to the first and second perturbation equations of *Julian and Gubbins* [1977], and that the third perturbation equation of *Julian and Gubbins* [1977] is identical to the constraint (23) with $\mathbf{g}$ and F given by (52).

9. Relation With the Perturbation Equations in Ray-Centered Coordinates

The perturbation equations in ray-centered coordinates are derived in a Hamiltonian formalism by *Farra and Madariaga* [1987] and in a Lagrangian formalism by *Snieder and Sambridge* [1992]. It should be noted that in the derivation of the latter paper the component of the acceleration along the reference curve ($-u_0(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_1)\dot{\mathbf{r}}_0$) is not eliminated. This can be seen in equation (32) of *Snieder and Sambridge* [1992] where every term is perpendicular to the reference curve. The requirement that the ray perturbation is perpendicular to the reference curve implies, according to equation (26) of *Snieder and Sambridge* [1992], that $(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) = -1/u_0 (\nabla u_0 \cdot \mathbf{r}_1)$. With (10) and (23) this corresponds to the choice:

$$\mathbf{g} = -\nabla u_0 \qquad F = 0 \ . \quad (54)$$

Since the stretch factor (54) has no free parameters, the associated mapping can two-point ray tracing problems only be realized for a restricted class of perturbations. This is reflected by the fact that the requirement that the perturbation is perpendicular to the reference curve implies that the end-point perturbations are also perpendicular to the reference curve. This is an undesirable restriction associated with ray-centered coordinates.

One can develop a three-dimensional equation for the ray perturbation by incorporating the stretch factor associated with ray-centered coordinates in the three-dimensional perturbation equations of this paper. One can in fact generalize the stretch factor (54) for ray-centered coordinates by using $\mathbf{g} = -\nabla u_0$ and specifying $F(s_0)$. This choice can be inserted in either (25), (36), or (37). In this way, one effectively prescribes the projection of the perturbation $\mathbf{r}_1$ on the reference curve. This can be seen by inserting the choice $\mathbf{g} = -\nabla u_0$ in (23):

$$u_0(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) = -(\nabla u_0 \cdot \mathbf{r}_1) + F(s_0) \ . \quad (55)$$

The ∇u_0 term is to order ε equal to $d(u_0\dot{\mathbf{r}}_0)/ds_0$; see equation (17). Using this to eliminate the ∇u_0 term in (55) gives

$$\frac{d}{ds_0}\left(u_0(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1) \right) = F(s_0) \ , \quad (56)$$

which can be integrated to give

$$u_0(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)(s_0) = u_0(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)(0) + \int_0^{s_0} F(s')\, ds' \ . \quad (57)$$

The left-hand side of this expression is the projection of the ray perturbation on the reference curve. When the ray perturbation is for a single point perpendicular to the reference curve ($(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1) = 0$ for some value s_0), one finds for the mapping (54) implied for ray centered coordinates from (56) that $(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1) = 0$ everywhere along the curve. This is the condition that the ray perturbation is perpendicular to the reference curve. When using ray centered coordinates, the ray perturbation at the end-points is necessarily perpendicular to the reference curve. Allowing a nonzero value for F in (57) makes it possible to accommodate end-point perturbations in different directions, which is crucial for the application to source relocation problems.

10. Relation With Hamiltonian Formalisms

In a number of papers, ray perturbation theory was analized using a Hamiltonian formulation of the ray tracing problem [*Chapman*, 1985; *Farra and Madariaga*, 1987; *Virieux et al.*, 1988; *Virieux*, 1991; *Farra*, 1992]. In *Farra and Madariaga* [1987], ray-centered coordinates were employed. The relation of their work with ray perturbation theory based on a Lagrangian formulation is discussed in *Snieder and Sambridge* [1992]. In the other papers, the ray perturbation was treated using three-dimensional Cartesian vectors. In these papers paraxial rays, slowness perturbations, and true ray bending were treated separately. The results of this paper imply that to first order in the ray update and to second order in the travel time, these effects can be treated simultaneously. The role of the stretch factor in the Hamiltonian formalisms is illustrated here for the Hamiltonian theory of *Farra* [1992] for pure ray bending. This means that the slowness is fixed ($u_1 = 0$).

From expressions (5) and (10) of *Farra* [1992] it follows that the perturbation in the position ($\Delta\mathbf{x}$) and the conjugate momentum ($\Delta\mathbf{p}$) satisfy the following system of first order equations:

$$\frac{d\Delta\mathbf{x}}{d\tau} = \Delta\mathbf{p} + \mathbf{p}_0 - \frac{d\mathbf{x}_0}{d\tau} \ , \qquad (58)$$

$$\frac{d\Delta\mathbf{p}}{d\tau} = \frac{1}{2}\nabla\nabla u_0^2 \cdot \Delta\mathbf{x} + \frac{1}{2}\nabla u_0^2 - \frac{d\mathbf{p}_0}{d\tau} \ . \qquad (59)$$

The independent parameter τ is defined as in *Virieux et al.* [1988]

$$\frac{d}{d\tau} = u\frac{d}{ds} = u_0\frac{d}{ds} \ . \qquad (60)$$

The reason for the restriction that the slowness is fixed ($u_1 = 0$) is that slowness enters the definition of the independent parameter (60). When the slowness is perturbed the identification between τ and the independent parameter s_0 of this paper is non-trivial because the last identity in equation (60) is not valid when $u_1 \neq 0$.

In order to make the connection with the results of this paper one should make the following change of notation: $\mathbf{x}_0 \rightarrow \mathbf{r}_0$ and $\Delta\mathbf{x} \rightarrow \varepsilon\mathbf{r}_1$. Furthermore, it follows from equation (2) of *Farra* [1992] and the choice of her Hamiltonian that

$$\mathbf{p}_0 = \frac{d\mathbf{x}_0}{d\tau} \rightarrow u_0\frac{d\mathbf{r}_0}{ds_0} \ , \qquad (61)$$

where (60) has been used in the last identity. Because of (61), the inhomogeneous term in (58) vanishes. For the inhomogeneous term in (59) one can use that

$$\frac{d\mathbf{p}_0}{d\tau} = u_0\frac{d}{ds_0}\left(u_0\frac{d\mathbf{r}_0}{ds_0}\right) , \qquad (62)$$

where (60) and (61) are used for the last identity. It follows from (58) that

$$\Delta\mathbf{p} = \frac{d\Delta\mathbf{x}}{d\tau} = \varepsilon u_0\frac{d\mathbf{r}_1}{ds_0} \ . \qquad (63)$$

Inserting this in (59) and using (60), (62) and (2) we obtain the appropriate ray perturbation equation

$$\frac{d}{ds_0}\left(u_0\frac{d\mathbf{r}_1}{ds_0}\right) - \frac{1}{2u_0}\nabla\nabla u_0^2 \cdot \mathbf{r}_1 = \frac{1}{\varepsilon}\left(\nabla u_0 - \frac{d}{ds_0}\left(u_0\frac{d\mathbf{r}_0}{ds_0}\right)\right) = \mathbf{R}_b \qquad (64)$$

This is equivalent to equation (31) for the special case $\mathbf{R}_1 = 0$ and $C = 0$. (The reason that there is no $\mathbf{R}_1$ term here is that *Farra* [1992] considers pure bending only, i.e., $u_1 = 0$ and hence $\mathbf{R}_1 = 0$.) This suggests that in the theory of *Farra* [1992] the stretch factor is given by (23) with $\mathbf{g} = \nabla u_0$, and indeed it can be shown explicitly that this is the case.

In Hamiltonian ray perturbation theory, the mapping from the reference curve to the perturbed curve is by definition specified by the fact that the original and the perturbed points correspond to the same value of the independent parameter τ. With (60) this implies that the mapping satisfies

$$\frac{ds}{u_0(\mathbf{r}_0 + \varepsilon\mathbf{r}_1)} = \frac{ds_0}{u_0(\mathbf{r}_0)} \ . \qquad (65)$$

From this expression it follows with (10) and a Taylor expansion of $u_0(\mathbf{r}_0 + \varepsilon\mathbf{r}_1)$ that

$$1 + \varepsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) = \frac{\partial s}{\partial s_0} = 1 + \frac{\varepsilon}{u_0}(\mathbf{r}_1 \cdot \nabla u_0) \ . \qquad (66)$$

A comparison of this expression with (23) implies that

$$\mathbf{g} = \nabla u_0 \qquad F = 0 \ . \qquad (67)$$

It is interesting to note that this stretch factor has exactly the opposite sign from the stretch factor used in ray-centered coordinates; see (54).

One can show that the stretch factor (66) implies that the perturbation of the Hamiltonian vanishes ($\Delta H = 0$), where the Hamiltonian is given by

$$H = \frac{1}{2}\left(p^2 - u^2\right) \ . \qquad (68)$$

Expressing the relation $\Delta H = 0$ in the perturbations $\Delta\mathbf{x}$ and $\Delta\mathbf{p}$ one obtains

$$\Delta H = \mathbf{p}_0 \cdot \Delta\mathbf{p} - \frac{1}{2}\nabla u_0^2 \cdot \Delta\mathbf{x} \ . \qquad (69)$$

With (61), (63) and the identification $\Delta\mathbf{x} \rightarrow \varepsilon\mathbf{r}_1$ this constraint can be written as

$$\Delta H = \varepsilon\left(u_0^2(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) - u_0(\nabla u_0 \cdot \mathbf{r}_1)\right) \ . \qquad (70)$$

Because of (66) the right-hand side vanishes:

$$\Delta H = 0 \ . \qquad (71)$$

Farra [1992] remarks that when the relation $\Delta H = 0$ is satisfied at the source, it is satisfied everywhere along the ray. This is consistent with expression (29) of this paper. The factor u_0 in (29) corresponds to the fact that (70) and the stretch factor (23) differ with a factor u_0.

In the Hamiltonian theory of this section, the stretch factor does not contain a free parameter, see (67). For initial value problems this is no problem. However, when applying the theory to two-point ray tracing problems the absence of a tuning parameter leads to an inconsistent system of equations. As shown in sections 11 and 12, this may lead to erroneous solutions.

11. Example 1

As an example of the role of a tuning parameter in the stretch factor, consider the problem of two-point ray tracing in a homogeneous reference medium ($u_0 = const$). Let the reference curve be a true ray (a straight line). Let the end-points perturbations be given by (3). The true perturbed ray is a straight line joining the perturbed end-points. In this example, $\mathbf{g} = \nabla u_0 = 0$ and $F = u_0 C$ is used. Note the special case $C = 0$ corresponds to the mapping employed in Hamiltonian ray perturbation theory.

For this particular problem, equation (31) governing the ray perturbation is given by

$$\ddot{\mathbf{r}}_1 = 0 \ . \tag{72}$$

The solution subject to the end-point perturbations (3) is given by

$$\mathbf{r}_1(s_0) = \mathbf{a} + (\ \mathbf{b} - \mathbf{a}\ )\ \frac{s_0}{S_0} \ . \tag{73}$$

The stretch factor for this problem follows by differentiation

$$(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1\ ) = \frac{1}{S_0}\ (\ \mathbf{b} - \mathbf{a}\ ) \cdot \dot{\mathbf{r}}_0 \ . \tag{74}$$

Inserting this relation in the stretch factor constraint (23) with the conditions $\mathbf{g} = \nabla u_0 = 0$ and $F = u_0 C$, one finds that

$$C = \frac{1}{S_0}\ (\ \mathbf{b} - \mathbf{a}\ ) \cdot \dot{\mathbf{r}}_0 \ . \tag{75}$$

This solution implies that C must be nonzero whenever the projection of the change in the ray length on the reference curve ($(\ \mathbf{b} - \mathbf{a}\ ) \cdot \dot{\mathbf{r}}_0$) is nonzero. The reason that C must be nonzero for this problem is related to the fact that the ray changes length under the perturbation of the end-points. Using the expressions (11) and (23) for $\mathbf{g} = 0$ and $F = u_0 C$, one finds that the length S of the perturbed ray is given by

$$S \equiv s(S_0) = (\ 1 + \varepsilon C\ )\ S_0 \ . \tag{76}$$

Hence the change in the ray length is accommodated by the tuning parameter C. One can readily verify that the relative change in the arc length is indeed given, to first order, by (75).

The procedure followed in this example is typical for finding the tuning parameter when solving (25) analytically. In general one can determine the unknown tuning parameter by inserting the solution of (25) in expression (23) for the stretch factor. Note that expression (29) implies that one only needs to do this at a single point along the ray, because if (23) is satisfied at a single point along the curve it is satisfied everywhere along the curve. When using the expressions (36) or (37) for the ray perturbation, there is no need to evaluate the tuning parameter because this parameter has been eliminated from the perturbation equation.

12. Example 2

Consider a reference medium where the slowness squared is a linear function of the space variables:

$$u_0^2 = \Gamma \cdot \mathbf{r} + A \ . \tag{77}$$

The Hamiltonian perturbation equations permit solutions in closed form for such a reference medium [e.g., *Virieux et al.*, 1988; *Virieux*, 1991], which makes it attractive to parameterize the medium in triangular or tetrahedal cells with a slowness variation (77) within each cell.

For such a medium the mapping $\mathbf{g} = \nabla u_0$, $F = u_0 C$ leads to solutions in closed form. With (77) it follows that

$$\frac{2F}{u_0}\ \nabla u_0 = 2C \nabla u_0 = \frac{C}{u_0}\ \Gamma \ . \tag{78}$$

Hence equation (31) is for this special case

$$u_0 \frac{d}{ds_0}\left(\ u_0 \frac{d\mathbf{r}_1}{ds_0}\ \right) = C\Gamma \ . \tag{79}$$

As a first example, consider the situation that the reference ray is a true ray whose end-points are perturbed according to (3). This example is of relevance when the medium is parameterized in triangular or tetrahedral cells and where the intersection of the ray with the cell boundaries change. It is useful to define a new independent parameter w by

$$w(s_0) \equiv \int_0^{s_0} \frac{1}{u_0}\ ds_0 \quad \text{and} \quad W \equiv w(S_0) \ . \tag{80}$$

Because of (60) this parameter is the same as the independent parameter τ used by *Farra* [1992]. With this new parameter, (79) is given by

$$\frac{d^2\mathbf{r}_1}{dw^2} = C\Gamma \ . \tag{81}$$

The solution of this equation subject to the boundary conditions (3) is given by

$$\mathbf{r}_1(w) = \mathbf{a} + (\ \mathbf{b} - \mathbf{a}\ )\ \frac{w}{W} - \frac{1}{2}\ C\Gamma\ w(W - w) \ . \tag{82}$$

At this point, the tuning parameter C is not yet determined. To determine this constant, the stretch factor for the solution (82) is needed

$$u_0(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1\ ) = (\dot{\mathbf{r}}_0 \cdot \frac{d\mathbf{r}_1}{dw})$$

$$= \frac{1}{W}\ (\ \mathbf{b} - \mathbf{a}\ ) \cdot \dot{\mathbf{r}}_0 - \frac{1}{2} C(\Gamma \cdot \dot{\mathbf{r}}_0)\ (\ W - 2w\ ) \ . \tag{83}$$

Inserting this stretch factor at the source position ($w = 0$) in (23) gives an expression that can be solved for C:

$$C = \left(\frac{1}{W}\ (\mathbf{b} - \mathbf{a}) \cdot \dot{\mathbf{r}}_0 - \frac{1}{2u_0}\ (\Gamma \cdot \mathbf{a})\right) \Big/ \left(u_0 + \frac{1}{2}\ (\Gamma \cdot \dot{\mathbf{r}}_0)W\right) \tag{84}$$

In general, C will be nonzero; setting the parameter C a priori to zero leads to an erroneous solution. As an example, let the reference slowness depend only on depth. Consider a reference ray starting and ending at the surface $z = 0$, and let the end-points of the ray be perturbed in opposite directions $\mathbf{a} = -\mathbf{b}$ along the surface (see Figure 2). For this situation the depth of the turning point of the true ray increases when the source receiver separation is increased. The ray perturbation predicted by current Hamiltonian formalisms are given by (82), with the constant C equal to zero:

$$\mathbf{r}_1^H = \mathbf{a} + (\mathbf{b} - \mathbf{a})\frac{w}{W} \ . \quad (85)$$

This solution is shown by the dashed line in Figure 2. Since the vector $\mathbf{r}_1^H$ in (85) has a vanishing component in the z direction, the turning point of the ray is not moved to greater depths when the source-receiver separation is increased.

In the full solution (82), the last term gives a nonzero vertical component to the ray perturbation. This solution is shown as the dashed line in Figure 3. The perturbed ray (82) approximates the true ray quite well. Note that the vertical component of the ray perturbation is given by $-1/2\, C\, \Gamma\, w(W - w)$, it is only nonzero when C is nonzero. One must incorporate this tuning factor in order to get a physically realistic ray perturbation with an increased turning point depth.

One can show that when the stretch factor is fixed a priori, the solution that is obtained is not only physically erroneous, but it is also inconsistent with the employed stretch factor. This is due to the fact that in this situation the problem is overdetermined; see section 4. Consider the solution $\mathbf{r}_1^H$ of equation (85) obtained for the fixed values $\mathbf{g} = \nabla u_0$, $F = u_0 C = 0$. It follows by direct differentiation that this solution satisfies

$$u_0(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1^H) = \frac{1}{W}(\mathbf{b} - \mathbf{a}) \cdot \dot{\mathbf{r}}_0 \ . \quad (86)$$

This solution also satisfies at the source ($w = 0$):

$$(\mathbf{g} \cdot \mathbf{r}_1^H) + F = (\nabla u_0 \cdot \mathbf{a}) \ . \quad (87)$$

Since the right-hand sides of (86) and (87) are in general different, the solution $\mathbf{r}_1^H$ does not satisfy (23) which defines the mapping. This implies that the employed equations are inconsistent, which is a consequence of the fact that the two-point ray tracing problem is overdetermined when the stretch factor is fixed a priori. In the analysis leading to equation (84) for the tuning parameter this problem did not occur because C was determined a posteriori by the condition (23) for the stretch factor.

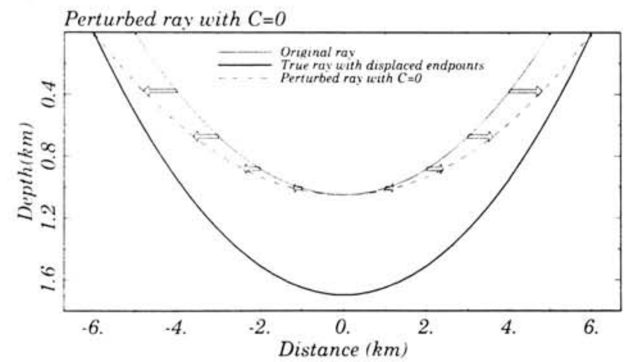

Fig. 2. Perturbation problem where the source and receiver are moved away from each other along the surface. The original ray is shown by a thin solid line; the true ray with perturbed end-points by the thick solid line. The (erroneous) ray perturbation (85) is shown by a dashed line. The mapping is defined by the arrows.

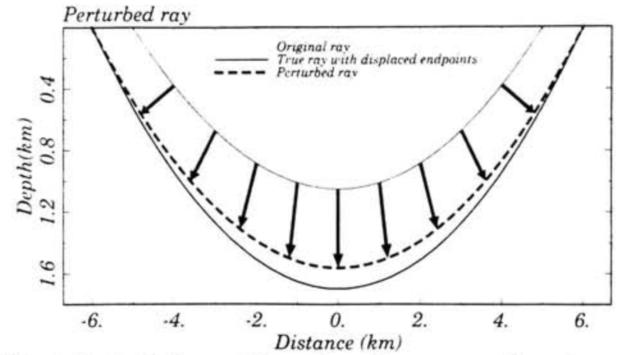

Fig. 3. Perturbation problem where the source and receiver are moved away from each other along the surface. The original ray is shown by a thin solid line; the true ray with perturbed end-points by the thick solid line. The ray perturbation (82) is shown by a dashed line. The mapping is defined by the arrows.

As a second example for the reference slowness (77), consider the perturbation problem where the end-points are fixed ($\mathbf{r}_1(0) = \mathbf{r}_1(W) = 0$), but where a forcing term $\mathbf{R}$ is present. Equation (31) for the ray perturbation is for this example:

$$\frac{d^2\mathbf{r}_1}{dw^2} = C\Gamma + u_0\mathbf{R} \ , \quad (88)$$

where $\mathbf{R}$ can be either $\mathbf{R}_b$ or $\mathbf{R}_1$ or a combination of the two. The solution of (88) is given by

$$\mathbf{r}_1(w) = \int_0^W G(w, w')u_0(\mathbf{r}_0(w'))\mathbf{R}(w')dw' - \frac{1}{2}C\Gamma w(W - w) \quad (89)$$

with the Green's function $G(w, w')$ given by

$$G(w, w') = -\frac{(W - w_>)w_<}{W}$$

$$w_> \equiv \max(w, w') \qquad w_< \equiv \min(w, w') \quad (90)$$

Inserting this solution at the source ($w = 0$) in the constraint (23) gives

$$C = -\dot{\mathbf{r}}_0 \cdot \int_0^W \frac{W - w'}{W} u_0(\mathbf{r}_0(w'))\, \mathbf{R}(w')\, dw' \,/\, (u_0 + \frac{1}{2}(\dot{\mathbf{r}}_0 \cdot \Gamma)W) \quad (91)$$

This means that also for this example the tuning parameter C is in general nonzero. Since the theory for $C = 0$ reduces to existing Hamiltonian ray perturbation theory this means that the application of Hamiltonian ray perturbation theory to two-point boundary value problems also leads to erroneous results when the slowness is perturbed or when a ray estimate is refined (ray bending).

13. DISCUSSION

The theory of this paper provides a natural explanation for the abundance of different perturbation theories for the ray position. The specification of the stretch factor in

(23) by the choice of g and F makes it possible to tailor the employed mapping to one's needs. For analytical derivations, equation (25) for the ray perturbation is most useful. In numerical applications to two-point ray tracing problems the necessicity of a tuning constant in F makes it preferable to use equations (36) or (37) for the ray perturbation.

For initial value ray tracing, one can use any choice for g and F. For two-point ray tracing problems one must incorporate a tuning constant in F in order to ensure consistency between the end-point conditions and the employed mapping. This tuning parameter prevents the system of equations from being overdetermined. The second order travel time perturbation can be computed at little additional cost by a single integration along the reference ray; see equation (45). For some analytical purposes, the equivalent equation (46) may be useful.

In previous ray perturbation theories, the stretch factor was defined (usually implicitly) from the onset. The freedom one has with the theory of this paper to choose the stretch factor explicitly is very useful because the equation for the ray perturbation has different analytical solutions for different stretch factors. For example, for the stretch factor of *Julian and Gubbins* [1977] given by equation (52) of this paper, equation (25) has analytical solutions for the ray perturbation when ∇u_0 is constant. The equations of *Snieder and Sambridge* [1992] permit analytical solutions when $\nabla(1/u_0)$ is constant. For the stretch factor (67), which is used in several formulations of Hamiltonian ray perturbation theory, analytical solutions exist when ∇u_0^2 is constant (see section 12). The stretch factor (54) permits analytical solutions when $(1/u_0)\nabla u_0$ is constant. These different analytical solutions may be very useful when one parameterizes the medium in terms of cells with a given behavior of the slowness variation. Analytical solutions for the ray perturbation can then be used for a variety of different slowness variations within each cell.

The examples of section 12 show that the application of existing Hamiltonian ray perturbation theory to two-point ray tracing problems leads in general to erroneous results. This does not mean that Hamiltonian ray perturbation theory is not correct. The deficiency of this theory for this particular example is caused by the fact that the Hamiltonian ray perturbation theory is formulated for intitial value problems. When being applied to to two-point boundary value problems, it must be extended to include nonzero values of the tuning parameter C. For applications of Hamiltonian ray perturbation theory where the slowness is parameterized in triangular or tetrahedral cells, one can express the ray perturbation within a cell analytically given the intersection points of the ray with the cell boundary. This leads to a two-point ray tracing problem within every cell. In those applications one should use the solution (82) rather than (85).

APPENDIX A

When deriving an expression for the first order ray perturbation from the equation of kinematic ray tracing (5), it is necessary to convert the derivatives along the perturbed curve to the derivatives along the unperturbed curve using

$$\frac{d}{ds} = \frac{\partial s_0}{\partial s}\frac{d}{ds_0} = (1 - \varepsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1))\frac{d}{ds_0} , \tag{A1}$$

where equation (9) is used for the last identity. Inserting this result and the perturbation expansions (4), (12), and (13) for $\mathbf{r}$, $u(\mathbf{r})$, and $\nabla u(\mathbf{r})$ in equation (5), one obtains up to order ε

$$(1 - \varepsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)) \times$$

$$\times \frac{d}{ds_0}\left\{ (u_0 + \varepsilon(u_1 + \mathbf{r}_1 \cdot \nabla u_0))\left(1 - \varepsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)\right)\frac{d}{ds_0}(\mathbf{r}_0 + \varepsilon\mathbf{r}_1)\right\}$$

$$= \nabla u_0 + \varepsilon(\nabla u_1 + \mathbf{r}_1 \cdot \nabla\nabla u_0) . \tag{A2}$$

When multiplying the different terms in the left-hand side, contributions of order ε^2 and higher can be ignored for the derivation of the lowest order ray perturbation. Using this, one obtains to $O(\varepsilon)$:

$$\varepsilon\left\{ -(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_0)\right.$$

$$\left. + \frac{d}{ds_0}\left((u_1 + \mathbf{r}_1 \cdot \nabla u_0)\,\dot{\mathbf{r}}_0 - u_0(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)\dot{\mathbf{r}}_0 + u_0\dot{\mathbf{r}}_1 \right)\right\}$$

$$= \varepsilon(\nabla u_1 + \mathbf{r}_1 \cdot \nabla\nabla u_0) + \left(\nabla u_0 - \frac{d}{ds_0}(u_0\dot{\mathbf{r}}_0)\right) . \tag{A3}$$

According to equation (2), the last term in this expression is equal to $\varepsilon\mathbf{R}_b$. Using this result, all terms in (A3) are proportional to ε, so that the complete expression can be divided by ε. Carrying out the differentiations with respect to s_0 and using the identity $d\nabla u_0 / ds_0 = (\dot{\mathbf{r}}_0 \cdot \nabla\nabla u_0)$, equation (A3) can after some algebra be written as

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) - u_0\dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{r}}_1) + \dot{\mathbf{r}}_1 \cdot \left\{ \nabla u_0 - u_0\ddot{\mathbf{r}}_0 - \frac{du_0}{ds_0}\dot{\mathbf{r}}_0 \right\}\dot{\mathbf{r}}_0$$

$$-(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1)\left[\frac{du_0}{ds_0}\dot{\mathbf{r}}_0 + 2u_0\ddot{\mathbf{r}}_0 \right] + (\mathbf{r}_1\dot{\mathbf{r}}_0 : \nabla\nabla u_0)\dot{\mathbf{r}}_0 + (\mathbf{r}_1 \cdot \nabla u_0)\ddot{\mathbf{r}}_0$$

$$-(\mathbf{r}_1 \cdot \nabla\nabla u_0) = \nabla u_1 - \frac{d}{ds_0}(u_1\dot{\mathbf{r}}_0) + \mathbf{R}_b . \tag{A4}$$

Because of equation (2) the terms between curly brackets are equal to $\varepsilon\mathbf{R}_b$. They therefore give a contribution to higher order in ε and can be ignored. The terms between square brackets can with equation (2) be written as

$$2u_0\ddot{\mathbf{r}}_0 + \frac{du_0}{ds_0}\dot{\mathbf{r}}_0 = 2\varepsilon\mathbf{R}_b - (\dot{\mathbf{r}}_0 \cdot \nabla u_0)\dot{\mathbf{r}}_0 + 2\nabla u_0 , \tag{A5}$$

where it is assumed that $du_0 / ds_0 = (\dot{\mathbf{r}}_0 \cdot \nabla u_0)$. The first term on the right-hand side in (A5) is of order ε and can to leading order be neglected. Inserting the remaining terms of (A5) in equation (A4) and rewriting the term $(\mathbf{r}_1\dot{\mathbf{r}}_0 : \nabla\nabla u_0)\dot{\mathbf{r}}_0$ as $\dot{\mathbf{r}}_0(\dot{\mathbf{r}}_0 \cdot \nabla\nabla u_0) \cdot \mathbf{r}_1$ leads to equation (14).

APPENDIX B

In order to simplify the integral (44*b*) for the second order travel time perturbation, the u_1 terms in this integral are eliminated. Using the definition (15) and an integration by parts, one obtains

$$\int_0^{s_0} (\mathbf{r}_1\cdot\mathbf{R}_1)\,ds_0 = \int_0^{s_0} \left((\mathbf{r}_1\cdot\nabla u_1) + (\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0)u_1 \right) ds_0 - [u_1(\dot{\mathbf{r}}_0\cdot\mathbf{r}_1)]_0^{s_0} , \tag{B1}$$

so that the second order travel time perturbation can be written as

$$T_2 = \int_0^{s_0} \left\{ \frac{u_0}{2}(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_1) - \frac{u_0}{2}(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1)^2 + (\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1)(\mathbf{r}_1\cdot\nabla u_0) + \frac{1}{2}(\mathbf{r}_1\mathbf{r}_1\cdot\nabla\nabla u_0) + \mathbf{r}_1\cdot(\mathbf{R}_b+\mathbf{R}_1) \right\} ds_0 + [u_1(\dot{\mathbf{r}}_0\cdot\mathbf{r}_1)]_0^{s_0} \tag{B2}$$

This expression can be simplified further by dotting (14) on the left with $\mathbf{r}_1$ and integrating the result over the reference curve

$$\int_0^{s_0} \Big(\mathbf{r}_1\cdot\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) - u_0(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1) + \frac{du_0}{ds_0}(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) - 2(\mathbf{r}_1\cdot\nabla u_0)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) + (\mathbf{r}_1\cdot\ddot{\mathbf{r}}_0)(\nabla u_0\cdot\mathbf{r}_1) + (\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_0\,\mathbf{r}_1 : \nabla\nabla u_0) - (\mathbf{r}_1\mathbf{r}_1 : \nabla\nabla u_0) \Big) ds_0 = \int_0^{s_0} \mathbf{r}_1\cdot(\mathbf{R}_b+\mathbf{R}_1)\,ds_0 . \tag{B3}$$

Using an integration by parts the first term is given by

$$\int_0^{s_0} \mathbf{r}_1\cdot\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1)\,ds_0 = -\int_0^{s_0} u_0(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_1)\,ds_0 + [u_0(\dot{\mathbf{r}}_1\cdot\mathbf{r}_1)]_0^{s_0} \tag{B4}$$

The second term in (B3) can be integrated by parts to remove the $\ddot{\mathbf{r}}_1$ term; this gives

$$-\int_0^{s_0} u_0(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_0\cdot\ddot{\mathbf{r}}_1)\,ds_0 = \int_0^{s_0} \Big(\frac{du_0}{ds_0}(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) + u_0(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0)^2 + u_0(\mathbf{r}_1\cdot\ddot{\mathbf{r}}_0)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) + u_0(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\ddot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) \Big) ds_0 - \Big[u_0(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0) \Big]_0^{s_0} . \tag{B5}$$

The fifth term in (B3) can also be integrated by parts in order to remove the $\ddot{\mathbf{r}}_0$ term:

$$\int_0^{s_0} (\mathbf{r}_1\cdot\ddot{\mathbf{r}}_0)(\nabla u_0\cdot\mathbf{r}_1)\,ds_0 = -\int_0^{s_0} \Big\{ (\dot{\mathbf{r}}_1\cdot\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) + (\mathbf{r}_1\dot{\mathbf{r}}_0 : \nabla\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) + (\mathbf{r}_1\cdot\nabla u_0)(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0) \Big\} ds_0 + \Big[(\mathbf{r}_1\cdot\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) \Big]_0^{s_0} . \tag{B6}$$

Inserting (B4)-(B6) in (B3) gives

$$\int_0^{s_0} \mathbf{r}_1\cdot(\mathbf{R}_b+\mathbf{R}_1)\,ds_0 = \int_0^{s_0} \Big\{ -u_0(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_1) + u_0(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0)^2 - 2(\mathbf{r}_1\cdot\nabla u_0)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) - (\dot{\mathbf{r}}_1\cdot\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) - (\mathbf{r}_1\mathbf{r}_1 : \nabla\nabla u_0) + \Big((\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0)\mathbf{r}_1 + (\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)\dot{\mathbf{r}}_1 \Big)\cdot\Big(u_0\ddot{\mathbf{r}}_0 + \frac{du_0}{ds_0}\dot{\mathbf{r}}_0 - \nabla u_0 \Big) \Big\} ds_0 + \Big[u_0(\dot{\mathbf{r}}_1\cdot\mathbf{r}_1) + (\mathbf{r}_1\cdot\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) - u_0(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0) \Big]_0^{s_0} \tag{B7}$$

According to (17), the term $u_0\ddot{\mathbf{r}}_0 + (du_0\,/\,ds_0)\,\dot{\mathbf{r}}_0 - \nabla u_0$ is of order ε; hence

$$\int_0^{s_0} \mathbf{r}_1\cdot(\mathbf{R}_b+\mathbf{R}_1)\,ds_0 = \int_0^{s_0} \Big\{ -u_0(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_1) + u_0(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0)^2 - 2(\mathbf{r}_1\cdot\nabla u_0)(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) - (\dot{\mathbf{r}}_1\cdot\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) - (\mathbf{r}_1\mathbf{r}_1 : \nabla\nabla u_0) \Big\} ds_0 + \Big[u_0(\dot{\mathbf{r}}_1\cdot\mathbf{r}_1) + (\mathbf{r}_1\cdot\nabla u_0)(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0) - u_0(\mathbf{r}_1\cdot\dot{\mathbf{r}}_0)(\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_0) \Big]_0^{s_0} . \tag{B8}$$

This relation can be used in (B2) to eliminate the terms quadratic in $\mathbf{r}_1$ and $\dot{\mathbf{r}}_1$; this leads to (45).

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