THE APPLICABILITY OF RAY PERTURBATION THEORY TO MANTLE TOMOGRAPHY

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Abstract Ray perturbation theory is an efficient means of calculating raypaths and travel times in 3-D heterogeneous media accurate to first and second order respectively. We present results from using perturbation theory in 3-D quasi-random models of the earth’s mantle and in a subduction model obtained from ISC tomography. We show that the perturbation theory provides travel time estimates with errors smaller than 0.1 s for a very wide range of heterogeneities. The second order travel time correction also removes the implicit bias, towards late arrivals, present in all first order estimates. The computational efficiency of the theory is discussed and actual cpu times are given. It is concluded that ray perturbation theory is both accurate and efficient enough to provide a basis for non-linear tomography on data sets with the order of a million rays.

Introduction

The three most common types of seismic ray tracing problem are initial value, two-point, and parametric ray tracing. Methods for their efficient solution play an important role in both exploration seismology and global seismology. In each case the objective is to find the new raypath and travel time that results from changes in either the ray end point conditions, or the slowness field. If the changes are small then each problem can be addressed using ray perturbation theory. Recently Snieder & Sambridge (1992) (hereafter referred to as S&S) applied a perturbation analysis to the equation of kinematic ray tracing and found simple expressions for the ray deflection accurate to first-order, and the travel time accurate to second order. Perturbation theory for the ray deflection only, has been developed by a number of authors, e.g., Farra & Madariaga (1987) and Virieux (1991), and compared to numerical ray bending schemes by Snieder & Spencer (1992). Although several schemes exist, to date none of them have been applied to the type of complex 3-D problems encountered in mantle tomography and exploration seismics. As the perturbations in slowness (or end points) become large the theory will break down and the accuracy of the raypaths and travel times will decrease, therefore, to be of use in seismic tomography we need a detailed understanding of its efficiency and accuracy as a function of the heterogeneity of the media. In this paper we test the theory of S&S for two-point my tracing problems in pseudo-random 3-D models, and 2-D models of subduction zones obtained from ISC delay time tomography. In each case we examine the accuracy of both raypaths and travel times compared to full 3-D my tracing.

Perturbation theory for travel times and rays

From eqns. (79) of S&S the first order my deflections in ray centred co-ordinates, $(q_1, q_2)$, which result from a slowness perturbation $u (x)$ are found by solving the decoupled set of second order differential equations,

$$
\frac{d}{ds} (u_1 q_1) + p^2 \left( \frac{\partial}{\partial u} \left[ \frac{1}{u} \right] \right) q_1 = p \frac{\partial}{\partial u} \left[ \frac{u}{u} \right] - \sqrt{u_x^2 - p^2} \frac{\partial}{\partial u} \left[ \frac{u}{u} \right],
$$

where $q_1$ is the reference slowness field, $p$ and $s$ are the ray parameter and arc length along the reference ray respectively. Eqns. (1) follow from the general eqn. (50) of S&S for the case when $u_1$ depends only on $x_1$ and the reference ray is in the $x_2$-$x_3$ plane. In this case $q_1$ describes my deflections in the plane of the reference ray and $q_2$ is in the perpendicular plane (see fig. 2. of S&S). For two-point my tracing both end points are fixed $q_1(0) = q_1(S_i) = 0, (i = 1, 2)$, where $S_i$ is the arc length of the reference ray. All quantities in (1) are evaluated along the reference ray and, after discretization, lead to a pair of tri-diagonal linear systems which are efficiently solved for $q_1(s)$ and $q_2(s)$.

The travel time of the perturbed ray in the perturbed medium, $T_{r^{+}}$, is given to second order by $T_{r^{+}} = T_r + T_1 + T_2$, where $T_1$ is the travel time of the reference ray, $T_2 = \int_{s_0}^{s_1} u ds$ follows from Fermat’s principle, and $T_2$ is the second order correction given by eqn.(61) of S&S,

$$
T_2 = \frac{1}{2} \int_{s_0}^{s_1} q_1 \left[ \frac{1}{u} \left( \frac{u}{u} \right) - \sqrt{u_x^2 - p^2} \frac{\partial}{\partial u} \left[ \frac{u}{u} \right] \right] + q_2 \frac{\partial}{\partial u} \left[ \frac{u}{u} \right]
$$

(2)

Once the ray deflection $(q_1, q_2)$ has been found from (1), all quantities in (2) are known and $T_2$ requires one extra integration along the reference ray.

Applicability as a function of rms and scale lengths in $u(x)$

The aim of this experiment is to examine the accuracy of ray perturbation theory compared to first order travel times (which form the basis of linearized tomography) as a function of rms and half-width of the slowness perturbation. Here the slowness perturbation consists of a 3-D quasi-random field with a Gaussian distribution. The field was generated in spherical co-ordinates using the method of Frankel & Clayton (1986) and added to the Jeffreys-Bullen reference model (see Bullen & Bolt, 1985). For convenience an earth-flattening transformation, EFT, (Germer & Mantshchishvich, 1966) was used to transform to a Cartesian co-ordinate frame (although in a real data case the spherical co-ordinate form of (1) & (2) given by S&S should be used). The statistical properties of the slowness perturbation were chosen to be the same at all radii in the spherical co-ordinate frame (which requires a simple correction to account for the distorting effect of the EFT).

S&S examined the accuracy of (1) and (2) for reference rays at three different ranges with a few values of $u_{r^{+}}$, and correlation lengths, $L_c$. In this paper we present results from 21 different rms slowness perturbations, 0.05% $\leq u_{r^{+}} \leq 3.00\%$, and 21 half-widths, equivalent to 100 $\leq L_c \leq 5000$ km in the spherical co-ordinate frame. For each $(u_{r^{+}}, L_c)$ pair all calculations were repeated in 10
The error in the first order travel time, $T_{i}^{\text{r}} = T_{i}^{\text{r}} + T_{i} - T_{i}^{\text{r}}$, is shown in fig. 1a, and the second order time, $T_{i}^{\text{r}} = T_{i}^{\text{r}} + T_{j}$, in fig. 1b. The irregularities in the contours are due to statistical fluctuations in the 3-D slowness fields. (For the most complex models the ray tracing algorithm failed to trace all 10 rays and we averaged over the successful ones.) The interval about 0.1 s is high-lighted to indicate the accuracy of the ISC travel times. For perturbation theory to be useful in mantle tomography the errors should be smaller than this value. From figs. 1a & 1b we see that all of the errors in $T_{i}$ are smaller than those in $T_{j}$ across the entire parameter range. The $T_{j}$ error remains below 0.1 s for a much wider range of heterogeneities than the $T_{i}$ error. Note that for a dominant upper mantle half-width of around 500 km (suggested by the work of Gudmundsson, Davies & Clayton, 1990) the $T_{j}$ error is less than 0.1 s for $u_{i},u_{j}$, as large as 3 %, while the $T_{i}$ error exceeds 0.1 s for models with $u_{i},u_{j}$ of only 0.8 %. The raypath diagrams represent the maximum error in the reference ray (fig. 2a), and the maximum error in the reference ray with the first order correction (fig. 2b), measured in the plane perpendicular to the reference ray (see fig. 5 of S&S). The white line represents the 70 km error level. Both sets of raypath errors increase dramatically above this value indicating that the theory is breaking down. Note that at a half-width of 500 km the ray path error from zeroth order theory exceeds 70 km for models

![Image of diagrams](https://example.com/diagram.png)

**Fig 1.** Travel time errors in a) first order, & b) second order approximations as a function of halfwidth and rms. ‘SH’ represents the position of seismic models based on spherical harmonic expansions to degree and order 6. ‘GDC’ represents the upper mantle model Gudmundsson et al. (1990). Other symbols explained in the text.

**Fig 2.** Maximum raypath errors between a) reference and true ray, & b) perturbed and true ray. (For all other features see fig. 1.)

**Fig 3.** Histograms of errors in first (dark) and second order (light) travel time approximations, for 500 rays in model A.
with $u_{v_{\text{sat}}} \geq 1.5\%$, but with the first order correction rays can be estimated in the same models with an error of between 5-10 km.

Figs. 1 & 2 show only the magnitude of the errors and not their sign. It is well known that for first arrivals $T_{1}^{*}$ is always positive, and that this systematic error can bias the models obtained from linearized tomography towards fast velocities. Fig. 3 contains a histogram of the $T_{1}^{* *}$ and $T_{2}^{* *}$ obtained for 500 independent slowness models with $L_{R} = 250$ km and $u_{v_{\text{sat}}} = 1.8\%$ (model A of fig. 1). The shorter columns are always plotted in the foreground. The positive bias in the first order travel time approximation (dark) is clearly seen in fig. 3, and is not present in the second order travel time errors (light). (Indeed there seems to be a slight preference in $T_{2}^{* *}$ towards negative errors.) This means that the use of the second order travel time correction remains important, even for models where the magnitude of $T_{1}^{* *}$ appears acceptably small.

Tomographic applications incorporating $T_{1}$ may well avoid the systematic bias in standard tomography. Furthermore outliers, which may have significant effects on tomographic reconstructions, are on average larger in the $T_{1}^{* *}$ distribution.

Application to a 2-D subduction model

The first example considers only random slowness fields and does not indicate the effect of organised structure on accuracy. The second example uses a 2-D slice through the Aegean subduction model EUR89A of Spakman et al. (1991). The main aim of this experiment is to determine whether the ray bending effect of the slab can be accurately handled by ray perturbation theory. The model has slowness perturbations ranging between ±3% and a well defined subduction zone of fast velocity (see fig. 4). A whole mantle model was generated by extending laterally to a distance of 91° and radially to the CMB. The lateral extension consists of an 11° section of a second model EUR89B (Spakman et al. 1991) repeated 6 times, with a smooth interpolation between sections. The major heterogeneity is restricted to the upper mantle by smoothly interpolating the slowness field to JB in depth. This ensures that our test model does not contain a large amount of structure outside the subduction zone which might contaminate the influence of the slab on the results, but has structure in the entire upper mantle with the same character as that in fig. 4. One should remember that the heterogeneity of our test model is likely to be less than that of the real earth and therefore our results may under-estimate the importance of ray perturbation theory.

Results are presented using sources placed along the slab at depths of 50, 120, 400 & 670 km (see fig. 4). True rays were traced from each source to 80 receivers spaced 100 km apart, $9^\circ \leq \Delta \leq 85^\circ$. Again an EFT was used and eqns. (1) & (2) were solved numerically for ray deflections & travel times, and compared to true rays calculated with the 3-D ray tracer. Travel time errors $T_{1}^{* *}$ and $T_{2}^{* *}$ are shown in fig. 5 as a function of epicentral range. The large $T_{1}$ error at point A in fig. 5, is due to rays leaving the slab immediately and hitting receivers located above the strong shallow heterogeneity (point $\alpha$ in fig. 4). The error is reduced significantly when $T_{1}$ is taken into account. A second error spike occurs at point B and is also due to effects outside the slab, i.e. rays bottoming in the region of
large vertical gradients around 700 km depth (point $\beta$ in fig. 4). If these complex regions result in multiple rays between source and receiver, then there is no guarantee that the perturbed ray is still the first arrival, and so the error spikes could be the result of two different phases being compared. This is also a problem in tomography based on standard first-order theory and is a consequence of the local perturbation approach. The error spike at B was also observed in experiments with sources outside of the slab (not shown), but it was not seen when using slab model EU R89B, which contains no anomaly at 700 km. As one would expect, the error spikes decrease in size as source depth increases (because the heterogeneous regions are sampled less). Rays in section B-C sample most of the slab. Here the error in $T_2$ is always larger than $T_1$ and always positive. For the two shallow sources $T_1 > 0.1$ s for some receivers at $\Delta > 45^\circ$, while the $T_2$ error remains less than 0.1 s for all sources. The importance of accurate ray paths in subduction zone tomography has been shown by VandeCar (1991). An examination of the ray deflections (not shown) also indicates that perturbation theory performs well for rays in section B-C by modelling both the amplitude and character of the true ray deflections, but does not match the amplitude of the true deflection for rays bottoming in the $\beta$ anomaly.

We conclude from this that my perturbation theory accounts to a large degree for the ray bending effects of slabs in seismic tomography. In our experiments the largest errors in both $T_1$ and $T_2$ are due to the anomalous regions outside of the slab. It turns out that the $\beta$ anomaly in fig. 4 is actually a tomographic artifact resulting from the use of an inadequate 1-D reference model. This emphasizes the importance of a suitable reference model in seismic tomography, a point also made by other authors, e.g. Van der Hilst & Spakman (1989).

Computational efficiency in 3-D models

At present full 3-D ray tracing is too expensive to use in tomography with data sets of the order of a million rays, and so the efficiency of ray perturbation theory is an important issue in determining its applicability to mantle tomography. The cost of evaluating $T_1$, $T_2$ and solving the O.D.E.'s in (1) depends linearly on the number of points used to discretize the reference ray, $n_{ref}$. We determine $n_{ref}$ by the number of equi-spaced points required to give an acceptable numerical error in the integration for $T_2$. In the 3-D example of model A in fig. 2, it was found that $T_1 (q_1, q_2)$ & $T_2$ could be calculated with a numerical error $< 0.006 s$ by using $n_{ref} = 129$, and this took an average of 0.033 s per ray on a Convex 210, which corresponds to a million rays in 9.3 hours of cpu. The $T_2$ integral amounted to $\approx 50\%$ of the total cpu, while calculations for $(q_1, q_2)$ and $T_2$ were completely vectorizable. The speed also scales linearly with the cost of evaluating $u_{s}(s)$. In this example 1-D cubic splines were used along the reference ray for both $u_{s}(s)$ and $u_{r}(s)$ and their derivatives. In tomographic applications further improvements in efficiency can be achieved by using analytical expressions for $u_{s}(r)$ and a cellular parameterisation for $u_{r}(r)$. It must be remembered that this cpu time applies only to a ray with $\Delta = 60^\circ$ and model A in fig. 1, although it suggests that ray perturbation theory will remain practical in mantle tomography with extremely large data sets.

Conclusion

We have compared the ray perturbation theory of S&$\S$ in quasi-random models of the earth's mantle over a wide range of rms slowness perturbations and length scales, and in organised subduction models. The results show that the theory gives accurate travel times and ray deflections across a much wider class of models than first order travel time theory. It also handles well the ray bending effects that typically occur in models of subduction. Since tomography tends to filter out complicated earth structure, the resulting models are likely to be too smooth and weak compared to the real earth. Therefore the level of heterogeneity in the real upper mantle may well compromise the first order travel time theory upon which linearized tomography is based. The ray perturbation theory of S&$\S$ remains accurate for a much wider class of heterogeneity, and provides a significantly more accurate theory for tomography.

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References


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