Wavefront Propagation in Layered Media using the Fast Marching Method

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Origin of the Fast Marching Method (FMM)

- First introduced by James Sethian (1996) as a general method for tracking the evolution of monotonically advancing interfaces.

Extracting the shape of an artery from an angiogram

Robotic navigation with constraints

- [http://www.math.berkeley.edu/~sethian](http://www.math.berkeley.edu/~sethian)
- The Eikonal equation $|\nabla_x T| = s(x)$ describes the monotonic propagation of a wavefront (interface) through a velocity field and therefore may be solved using the FMM.
Motivation

- In its original form, the FMM is only capable of finding the first-arriving traveltime field; later arriving phases cannot be found.

- There are many other grid-based methods designed for rapidly calculating the first-arrival traveltime field including:
  - Methods based on Vidale’s (1988,1990) finite difference solution of the Eikonal equation.
  - Schemes that propagate the first-arriving wavefront using Huygen’s principle.
  - Shortest Path Raytracing (SPR) which locates the path of minimum traveltime through a network.

- The FMM distinguishes itself for the following three reasons:
  1. Unconditional stability.
  2. Rapid computation. For $M$ nodes, the operation count is $O(M \log M)$.
  3. Potential to be modified to track later arriving phases.

- The unconditional stability of the scheme comes from properly addressing the development and propagation of gradient discontinuities in the evolving wavefront.

- The appropriate weak solution comes from solving the “viscous” version of the Eikonal equation:
  $$|\nabla_x T| = s(x) + \epsilon \nabla_x^2 T$$
  for $\epsilon \to 0$ (the viscous limit). This is equivalent to enforcing the entropy condition “once a point burns, it stays burnt”.

- Entropy satisfying weak solutions may be obtained by using upwind gradient operators which take into account the direction of flow of information.
Propagating Cosine Curve

Swallow tail $\varepsilon=0.005$ $\varepsilon=0$

Central Difference Approximations to Gradient

Exact $dt=0.005$ $dt=0.0005$

Upwind Entropy-Satisfying Approximations

Exact 20 points 100 points
**Implementation**

- The FMM systematically constructs traveltimes $T$ in a downwind fashion from known values upwind by employing a *narrow band* approach.

- A heap sorting algorithm is used to find the minimum traveltime point within the narrow band at each step → $O(M \log M)$ operations.
• A **First Order** accurate upwind difference scheme is given by:

\[
\left[ \max(D_{ijk}^x T, -D_{ijk}^+ x T, 0)^2 + \max(D_{ijk}^y T, -D_{ijk}^+ y T, 0)^2 + \max(D_{ijk}^z T, -D_{ijk}^+ z T, 0)^2 \right]^{\frac{1}{2}} = s_{ijk}
\]

where:

\[
D_{ijk}^{\pm x} T = \frac{T(x + \delta x) - T(x)}{\delta x}
\]

\[
D_{ijk}^{\pm y} T = \frac{T(x) - T(x - \delta x)}{\delta x}
\]

• A **Mixed Order** upwind difference scheme is given by:

\[
\left[ \max \left( \left( D_{ijk}^x T + s_{ijk}^{-x} \delta_x (D_{ijk}^{-x})^2 T \right), \left( D_{ijk}^+ x T - s_{ijk}^{+x} \delta_x (D_{ijk}^{+x})^2 T \right), 0 \right)^2 \right]^{\frac{1}{2}} + \\
\left[ \max \left( \left( D_{ijk}^{-y} T + s_{ijk}^{-y} \delta_y (D_{ijk}^{-y})^2 T \right), \left( D_{ijk}^+ y T - s_{ijk}^{+y} \delta_y (D_{ijk}^{+y})^2 T \right), 0 \right)^2 \right]^{\frac{1}{2}} + \\
\left[ \max \left( \left( D_{ijk}^{-z} T + s_{ijk}^{-z} \delta_z (D_{ijk}^{-z})^2 T \right), \left( D_{ijk}^+ z T - s_{ijk}^{+z} \delta_z (D_{ijk}^{+z})^2 T \right), 0 \right)^2 \right]^{\frac{1}{2}} = s_{ijk}
\]

where the switch function:

\[
s_{ijk}^{-x} = \begin{cases} 1 & (T_{i-2,j,k}, T_{i-1,j,k}) \text{ are known and } T_{i-2,j,k} \leq T_{i-1,j,k} \\ 0 & \text{otherwise} \end{cases}
\]

\[
s_{ijk}^{-x} = \begin{cases} 1 & (T_{i+2,j,k}, T_{i+1,j,k}) \text{ are known and } T_{i+2,j,k} \leq T_{i+1,j,k} \\ 0 & \text{otherwise} \end{cases}
\]

chooses the first-order scheme if the second-order scheme does not respect causality.
FMM Examples

- The following examples apply the FMM in velocity media parameterized in terms of bicubic B-spline functions on a regular grid.
The following two examples implicitly introduce layering to allow the calculation of refracted phases. Layer boundaries are described by cubic B-splines.
**Reflected phases with the FMM**

- We adapt the FMM to allow reflected phases to be tracked in layered media.

- A regular grid is retained to describe the velocity media, but a locally irregular mesh of triangles sutures the regular cells to the nodes of the irregular boundary.

- A first-order upwind scheme is used within the triangulated domain; elsewhere, the mixed order scheme is used.

- A two-stage process is used to track the reflected phases.
Examples

Incident Wavefront

Reflected Wavefront

Both Wavefronts
**Summary**

- The desirable properties of the FMM include:
  1. Rapid computation ($O(M \log M)$).
  2. Unconditional stability (arbitrary velocity variations).
  3. Potential to be modified to track later arriving phases (layered media).

- We have modified the FMM to calculate reflected phases by:
  - using an adaptive irregular triangular mesh to locally suture the regular velocity nodes to the irregular interface nodes.
  - applying a two-stage FMM which tracks the wavefront to the interface and then reinitializes it back to the surface by employing the interface traveltime surface as the narrow band.

- The new scheme was successfully applied to highly complex layered velocity media. Incident shocks were correctly reflected.

- Future development of the scheme is directed towards locating later arriving phases of any specified type.