PEAT8002 - SEISMOLOGY
Lecture 14: Free Oscillations I

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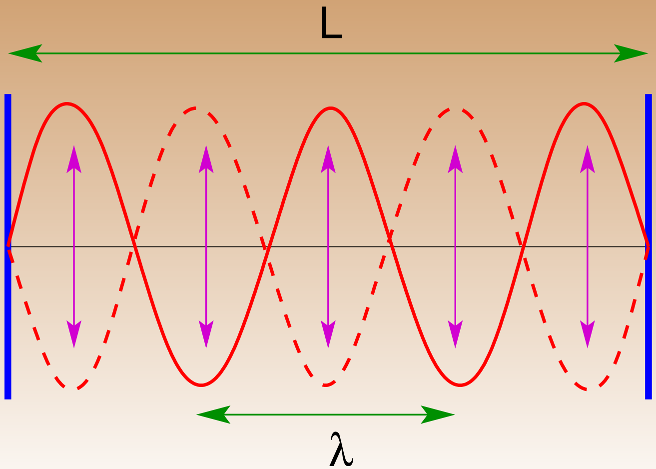
- Like a stretched string on a guitar, an elastic sphere supports standing waves. In seismology, these are known as free oscillations or normal modes, and are characterises by a discrete spectrum.
- Only specific frequencies are permitted, corresponding to integral numbers of wavelengths in the 3 orthogonal directions (r, θ, Φ).
- These oscillations correspond to standing surface waves of the longest possible wavelength and lowest frequency (periods up to about 1 hour).
- The longest period oscillations are only excited in a measurable way by the largest earthquakes, and it was not until the 1960 Chile earthquake that they were unambiguously identified on long-period seismograms.

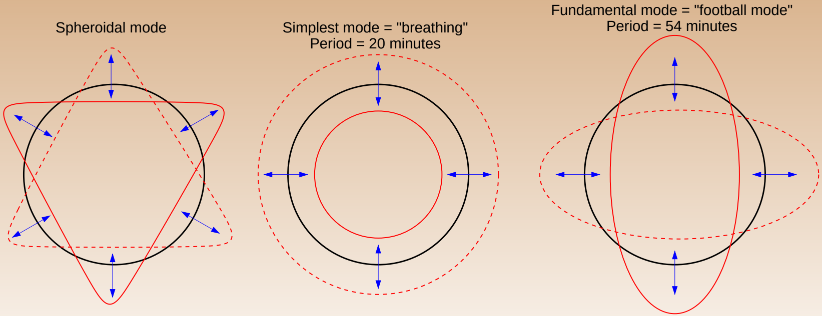
- The complete set of normal modes forms a basis for the description of any general elastic displacement that can occur within the Earth, and this property is used to calculate theoretical seismograms for long-period surface waves.
- Similar to the separation of Love and Rayleigh surface waves, the normal modes separate into two distinct types of oscillation: spheroidal, corresponding to standing Rayleigh waves, and toroidal, corresponding to standing Love waves.
- If we consider a piece of elastic string fixed at both ends a distance L apart, then standing waves only occur at discrete frequencies defined by:

$$\lambda = \frac{2L}{n} \qquad n = 1, 2, 3, ...$$

- In this case, $n = 1$ defines the fundamental mode.

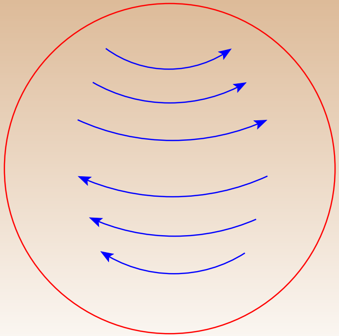


- The figure below demonstrates the concept of spheroidal modes. These are equivalent to standing Rayleigh waves with coupled P-SV motion.

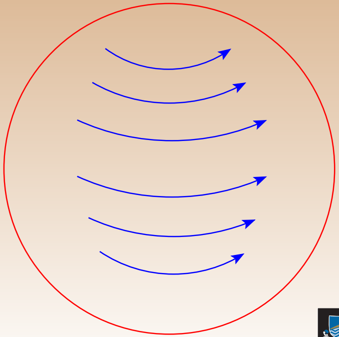


- The Figure below demonstrates the concept of torsional modes. The simplest torsional mode corresponds to a rotating Earth, which is of no interest. The next mode involves two hemispheres moving in opposite directions.

Torsional modes



Simplest torsional mode



- We previously used Helmholtz' theorem to show the separation of P and S waves:

$$\mathbf{u} = \nabla\Phi + \nabla \times \mathbf{\Psi}$$

- In the above equation, $\mathbf{u}$ is displacement, Φ is the scalar potential of the displacement field, and $\mathbf{\Psi}$ is the vector potential of the displacement field.
- This theorem can be slightly modified for use in spherical analysis, as we can separate the vector potential into the orthogonal radial and horizontal components:

$$\mathbf{\Psi} = T\hat{\mathbf{r}} + \nabla \times (S\hat{\mathbf{r}})$$

- Thus, the displacement can be written:

$$\mathbf{u} = \nabla\Phi + \nabla \times T\hat{\mathbf{r}} + \nabla \times \nabla \times (S\hat{\mathbf{r}})$$

- Now there are three scalar potentials, T, giving the torsional mode oscillations (SH type displacement), and Φ and S, giving the spheroidal mode oscillations (coupled P and SV displacements).
- From earlier, the Navier equation of motion was written as:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla\theta + \mu\nabla^2\mathbf{u} + \rho\nabla U$$

where ∇U is the gravitational acceleration.

- Since $\nabla\theta = \nabla(\nabla \cdot \mathbf{u})$ and $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$,

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u} + \rho\nabla U$$

- If we assume that periodic solutions exist with angular frequency ω, then

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = -\rho\omega^2\mathbf{u}$$

This occurs because if we consider harmonic oscillations described by $\mathbf{u} = A\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, then $\partial^2\mathbf{u}/\partial t^2 = -\omega^2\mathbf{u}$

- Substitution into the Navier equation gives:

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u} + \rho\nabla U = -\rho\omega^2\mathbf{u} \qquad (1)$$

- We now substitute $\mathbf{u} = \nabla\Phi + \nabla \times T\hat{\mathbf{r}} + \nabla \times \nabla \times (S\hat{\mathbf{r}})$ into equation (1).
- The separate components involving displacement can be considered as follows:

$$\nabla \cdot \mathbf{u} = \nabla \cdot \nabla\Phi + \nabla \cdot (\nabla \times T\hat{\mathbf{r}}) + \nabla \cdot (\nabla \times \nabla \times (S\hat{\mathbf{r}})) = \nabla^2\Phi$$

noting that the divergence of a curl is zero.

- The next term is:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{u} &= \nabla \times \nabla \times (\nabla \Phi) + (\nabla \times)^3 T\hat{\mathbf{r}} + (\nabla \times)^4 (S\hat{\mathbf{r}}) \\ &= (\nabla \times)^3 T\hat{\mathbf{r}} + (\nabla \times)^4 (S\hat{\mathbf{r}}) \end{aligned}$$

  noting that the curl of a gradient is zero.
- Substitution into Equation (1) therefore gives:

$$-\rho\omega^2 \mathbf{u} = (\lambda + 2\mu)\nabla(\nabla^2 \Phi) - \mu[(\nabla \times)^3 T\hat{\mathbf{r}} + (\nabla \times)^4 (S\hat{\mathbf{r}})] + \rho \nabla U$$

- If we now take the divergence $(\nabla \cdot)$ of this expression:

$$\begin{aligned} \nabla \cdot (-\rho\omega^2 \mathbf{u}) &= \nabla \cdot (\lambda + 2\mu)\nabla(\nabla^2 \Phi) \\ &\quad -\nabla \cdot \mu[(\nabla \times)^3 T\hat{\mathbf{r}} + (\nabla \times)^4 (S\hat{\mathbf{r}})] + \nabla \cdot \rho \nabla U \end{aligned}$$

- The first term $\nabla \cdot (-\rho\omega^2 \mathbf{u}) = -\rho\omega^2 \nabla^2 \Phi$ since $\nabla \cdot \mathbf{u} = \nabla^2 \Phi$.
- The third term $\nabla \cdot \mu[(\nabla\times)^3 T\hat{\mathbf{r}} + (\nabla\times)^4 (S\hat{\mathbf{r}})] = 0$ because the divergence of a curl is zero.
- The final term $\nabla \cdot \rho \nabla U = \rho \nabla^2 U \approx 0$, if we assume the second derivative of U is negligible (variations in body force ignored).
- Putting these terms together gives:

$$\boxed{\rho\omega^2 \nabla^2 \Phi + (\lambda + 2\mu) \nabla^4 \Phi = 0}$$

- The above equation is for a scalar potential Φ with P-type oscillation.

- If we now take the curl $(\nabla\times)$ of Equation (1),

$$\begin{aligned}\nabla\times(\rho\omega^2\mathbf{u}) &= \nabla\times(\mu\nabla\times\nabla\times\mathbf{u})-\nabla\times(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})\\ &\quad-\nabla\times(\rho\nabla U)\end{aligned}$$

  The last two terms are zero because the curl of a gradient is zero.

- Since $\nabla\times\mathbf{u}=\nabla\times\nabla\times T\hat{\mathbf{r}}+(\nabla\times)^3S\hat{\mathbf{r}}$, then

$$(\nabla\times)^3\mathbf{u}=(\nabla\times)^4T\hat{\mathbf{r}}+(\nabla\times)^5S\hat{\mathbf{r}}$$

- This allows us to form the following equation:

$$\rho\omega^2[(\nabla\times)^2T\hat{\mathbf{r}}+(\nabla\times)^3S\hat{\mathbf{r}}]=\mu(\nabla\times)^4T\hat{\mathbf{r}}+\mu(\nabla\times)^5S\hat{\mathbf{r}} \qquad (2)$$

- If we now take the dot product of Equation (2) with $\hat{\mathbf{r}}$, then

$$\rho\omega^2\hat{\mathbf{r}}\cdot[(\nabla\times)^2 T\hat{\mathbf{r}}+(\nabla\times)^3 S\hat{\mathbf{r}}] = \mu\hat{\mathbf{r}}\cdot(\nabla\times)^4 T\hat{\mathbf{r}}+\mu\hat{\mathbf{r}}\cdot(\nabla\times)^5 S\hat{\mathbf{r}}$$

- We now make use of the following results from vector algebra:

$$\hat{\mathbf{r}}\cdot\nabla\times(\nabla\times)^{2n}(\hat{\mathbf{r}}f) = 0, \qquad n>0$$

$$\hat{\mathbf{r}}\cdot\nabla\times\nabla\times(\hat{\mathbf{r}}f) = \frac{-1}{r^2}{\nabla_h}^2 f$$

$$\hat{\mathbf{r}}\cdot(\nabla\times)^{2n}(\hat{\mathbf{r}}f) = \frac{(-1)^2}{r}\nabla^{2(n-1)}{\nabla_h}^2\left(\frac{f}{r}\right)$$

where $f = f(r,\theta,\phi)$ is a scalar function and ${\nabla_h}^2$ is the horizontal component of ∇.

- It therefore follows that $\hat{\mathbf{r}} \cdot (\nabla\times)^3 S\hat{\mathbf{r}} = 0$ and $\hat{\mathbf{r}} \cdot (\nabla\times)^5 S\hat{\mathbf{r}} = 0$.
- The above equation then reduces to:

$$\frac{-\rho\omega^2}{r^2}{\nabla_h}^2 T = \frac{\mu}{r}\nabla^2{\nabla_h}^2\frac{T}{r}$$

- Noting that r is not operated on by ${\nabla_h}^2$, we can write this equation as

$$\boxed{\frac{\rho\omega^2}{r}{\nabla_h}^2\left(\frac{T}{r}\right) + \frac{\mu}{r}\nabla^2{\nabla_h}^2\left(\frac{T}{r}\right) = 0}$$

- The above equation is for a torsional potential (SH type oscillation).

- If we now take the curl of equation (2), then

$$\rho\omega^2[(\nabla\times)^3 T\hat{\mathbf{r}} + (\nabla\times)^4 S\hat{\mathbf{r}}] = \mu(\nabla\times)^5 T\hat{\mathbf{r}} + \mu(\nabla\times)^6 S\hat{\mathbf{r}}$$

  The terms involving T are equal to zero because they are premultiplied by $(\nabla\times)^n$ where n is an odd number.

- If we take the dot product of both sides of the equation with $\hat{\mathbf{r}}$, the the two terms for S become:

$$\hat{\mathbf{r}}\cdot(\nabla\times)^4 S\hat{\mathbf{r}} = \frac{1}{r}\nabla^2{\nabla_h}^2\left(\frac{S}{r}\right) \qquad \hat{\mathbf{r}}\cdot(\nabla\times)^6 S\hat{\mathbf{r}} = -\frac{1}{r}\nabla^4{\nabla_h}^2\left(\frac{S}{r}\right)$$

- We thus end with an equation for the spheroidal potential (SV oscillation):

$$\boxed{\rho\omega^2\nabla^2{\nabla_h}^2\left(\frac{S}{r}\right) + \mu\nabla^4{\nabla_h}^2\left(\frac{S}{r}\right) = 0}$$

- Thus, we have derived three equations:

$$\rho\omega^2\nabla^2\Phi + (\lambda + 2\mu)\nabla^4\Phi = 0 \qquad \text{(a)}$$

$$\frac{\rho\omega^2}{r}{\nabla_h}^2\left(\frac{T}{r}\right) + \frac{\mu}{r}\nabla^2{\nabla_h}^2\left(\frac{T}{r}\right) = 0 \qquad \text{(b)}$$

$$\rho\omega^2\nabla^2{\nabla_h}^2\left(\frac{S}{r}\right) + \mu\nabla^4{\nabla_h}^2\left(\frac{S}{r}\right) = 0 \qquad \text{(c)}$$

- Each of these equations may be written in the form:

$$\left(\frac{\omega}{c}\right)^2 f + \nabla^2 f = 0$$

which is often referred to as Helmholtz' equation.

- The Helmholtz' equation is the time-independent form of a wave equation, derived by assuming a separation of variables (in this case displacement **u**).
- In the case of Equation (a), c is the P-wave velocity, given by:

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \text{where} \qquad f = \nabla^2 \Phi$$

- In the case of Equation (b), c is the S-wave velocity, given by:

$$\beta = \sqrt{\frac{\mu}{\rho}} \qquad \text{where} \qquad f = \nabla_h{}^2 (T/r)$$

- In the case of Equation (c), c is the S-wave velocity, given by:

$$\beta = \sqrt{\frac{\mu}{\rho}} \qquad \text{where} \qquad f = \nabla^2 \nabla_h{}^2 (S/r)$$

- In order to solve the three Helmholtz equations, we need boundary conditions. Here, we can use the stress-free boundary conditions at the surface of the Earth $\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0$.
- As in the case of plane waves reflected from a free surface, the P-SV components must couple in order to satisfy the boundary conditions. The SH component is independent.
- Thus, the Torsional mode T solution is independent of the other two modes (implies long period standing Love waves).
- The Spheroidal mode solution required Φ and S to be solved simultaneously since P and SV are coupled.

- In the special case of a fluid sphere, T and S vanish (since $\mathbf{u} = \nabla\Phi$ in this case).
- Therefore, dilatation $\theta = \nabla^2\Phi$. Since the pressure $p = \sigma_{kk}/3$ and $\sigma_{ij} = \lambda\theta\delta_{ij}$, then

$$\theta = \frac{p}{\lambda} = \frac{p}{\kappa}$$

  where κ is the adiabatic bulk modulus.
- Thus, the pressure field of a free oscillation must directly satisfy Helmholtz' equation:

$$\left(\frac{\omega}{c}\right)^2 p + \nabla^2 p = 0$$

- From before, we have Helmholtz' equation written as

$$\left(\frac{\omega}{c}\right)^2 f + \nabla^2 f = 0 \qquad (3)$$

where $f = f(r, \theta, \phi)$. Solution in 3-D spherical geometry can be expressed in terms of spherical harmonics.

- Note that we have assumed an oscillatory time dependence ($\exp[i\omega t]$) in order to derive Equation (3). Solutions $f(r, \theta, \phi)$ to Equation (3) correspond to eigenfunctions. Each eigenfunction is associated with a specific value $(w/c)^2$, which is an eigenvalue.
- In order to solve for f, we seek separable solutions of the form:

$$f(r, \theta, \phi) = R(r)Q(\theta)F(\phi)$$

- The Laplacian operator in spherical coordinates is written:

$$\nabla^2 f = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right]$$

- If we substitute our expressions for f and $\nabla^2 f$ into Equation (3) we obtain:

$$\left(\frac{\omega}{c}\right)^2 RQF + \frac{1}{r^2}\left[QF\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{RF}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right) + \frac{RQ}{\sin^2\theta}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}\right] = 0$$

- Dividing through by f gives:

$$\frac{1}{Rr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{r^2 Q\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right)+\frac{1}{Fr^2\sin^2\theta}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}+\left(\frac{\omega}{c}\right)^2=0$$

- We can now separate variables by multiplying through by $r^2\sin^2\theta$:

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{\sin\theta}{Q}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right)+r^2\sin^2\theta\left(\frac{\omega}{c}\right)^2=-\frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}=m^2$$

where m^2 is a separation constant.

- The azimuthal (Φ-dependent) component of the solution is:

$$\frac{d^2F}{d\phi^2} = -m^2F$$

which has a solution of the form

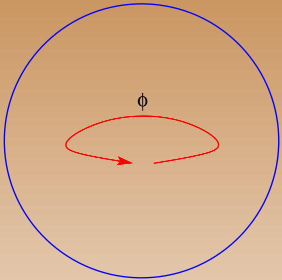
$$F(\phi) = A\exp[im\phi]$$

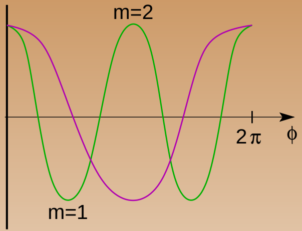
(this can be verified by substitution).

- For this solution to be valid, $F(\phi)$ must be continuous everywhere ($F(0) = F(2\pi)$), which means that $m = \pm 1, \pm 2, \pm 3, \ldots\ldots$
- Note that ϕ is the longitude variable, but the orientation of the pole in this coordinate system is arbitrary.

Solving Helmholtz' Equation

Azimuthal component of solution





- The remaining portion of the equation is:

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{\sin\theta}{Q}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right)$$
$$+r^2\sin^2\theta\left(\frac{\omega}{c}\right)^2=m^2$$

- If we now divide through by $\sin^2\theta$ to separate variables:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + r^2\left(\frac{\omega}{c}\right)^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{Q\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right) = K$$

  where K is a separation constant.
- We can now define a radial function:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left[\left(\frac{\omega}{c}\right)^2 - K\right]R = 0$$

  and a zonal function:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right) = Q\left[\frac{m^2}{\sin\theta} - K\sin\theta\right]$$

- We now examine solutions for the zonal function (where θ is the co-latitude, or $90^\circ -$latitude).
- First, we will look at the case where $m = 0$. The zonal function then becomes:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right) = -QK\sin\theta$$

- Application of the product rule gives:

$$\sin\theta\frac{\mathrm{d}^2Q}{\mathrm{d}\theta^2} + \cos\theta\frac{\mathrm{d}Q}{\mathrm{d}\theta} + QK\sin\theta = 0 \qquad (4)$$

- If we now let $x = \cos\theta$, so $\mathrm{d}x/\mathrm{d}\theta = -\sin\theta$, then

$$\frac{\mathrm{d}Q}{\mathrm{d}\theta} = \frac{\mathrm{d}Q}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{\mathrm{d}Q}{\mathrm{d}x}\sin\theta$$

- Taking the second derivative of Q:

$$\begin{aligned}\frac{d^2Q}{d\theta^2} &= -\frac{d}{d\theta}\left(\frac{dQ}{dx}\sin\theta\right)\\ &= -\frac{d}{dx}\left(\frac{dQ}{dx}\sin\theta\right)\frac{dx}{d\theta}\\ &= \left(\sin\theta\frac{d^2Q}{dx^2}+\frac{d(\sin\theta)}{d\theta}\frac{d\theta}{dx}\frac{dQ}{dx}\right)\sin\theta\end{aligned}$$

- Therefore:

$$\frac{d^2Q}{d\theta^2} = \sin^2\theta\frac{d^2Q}{dx^2} - \cos\theta\frac{dQ}{dx}$$

- Substitution in Equation (4) yields:

$$\sin\theta\left(\sin^2\theta\frac{d^2Q}{dx^2}-\cos\theta\frac{dQ}{dx}\right)-\cos\theta\sin\theta\frac{dQ}{dx}+KQ\sin\theta=0$$

- Dividing through by $\sin\theta$, and substituting in $x=\cos\theta$ gives:

$$\boxed{(1-x^2)\frac{d^2Q}{dx^2}-2x\frac{dQ}{dx}+KQ=0}$$

- This final expression is known as the Legendre equation.
- For certain discrete values of K, there are non-singular solutions in the interval $-1\leq x\leq 1$ (corresponding to $0\leq\theta\leq\pi$). These values are:

$$K=l(l+1),\qquad l=1,2,\dots.$$

- The solution of the Legendre equation can be written as an l^{th} order polynomial.
- A general expression for this solution is given by Rodrigues' formula:

$$P_l(x) = \frac{1}{2^l l!}\left[\frac{d^l}{dx^l}(x^2 - 1)^l\right]$$

- The first few Legendre polynomials are:

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

- If we now consider the case with $m \geq 0$, then the modified Legendre equation has solutions only when $K = l(l+1)$ as before, but in addition, $|m| \leq l$, so there are $2l+1$ possible values of m. The solutions in this case are:

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\left[\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l\right]$$

- For negative indices,

$$P_l^{-m}(x) = (-1)^m \left[\frac{(l-m)!}{(l+m)!}\right] P_l^m(x)$$

- l is sometimes referred to as the zonal angular order number, and m the tesseral (or azimuthal) angular order number.

- Thus, $|m|$ is the number of nodal planes that pass through the poles, and $l - |m|$ is the number of nodal surfaces parallel to the equator.