

PEAT8002 - SEISMOLOGY

Lecture 15: Free Oscillations II

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Free oscillations II

Normalized surface harmonics

- At the end of the last lecture, we showed that the (r, θ) components of free oscillation (the spheroidal mode) on a homogeneous sphere were described by the Legendre equation.
- Here, we briefly discuss the spherical harmonic representation of spheroidal modes.
- The normalised surface harmonic defined for $l \geq 0$, $-l \leq m \leq l$ is given by:

$$Y_l^m(\theta, \phi) = \hat{K}_l^m F(\phi) P_l^m(\cos \theta) = K_l^m P_l^m(\cos \theta) \exp[im\phi]$$

Recalling that $F(\phi) = A \exp[im\phi]$, and using $K_l^m = A \hat{K}_l^m$.

- The normalisation constant K_l^m is chosen so that the integral over the surface of the sphere of each harmonic times itself is unity, i.e.

$$\int_0^{2\pi} d\phi \int_0^\pi [Y_l^m(\theta, \phi)]^2 \sin \theta d\theta = 1$$

Free oscillations II

Normalized surface harmonics

- It turns out that for this condition to be satisfied,

$$K_l^m = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}$$

- The spherical harmonics form a complete orthonormal basis on the sphere:

$$\int_0^{2\pi} d\phi \int_0^\pi [Y_l^m(\theta, \phi)][Y_{l'}^{m'}(\theta, \phi)] \sin\theta d\theta = \delta_{ll'} \delta_{mm'}$$

- Any function defined on the surface of the sphere may be expressed as a sum of spherical harmonics (analogous to Fourier series in 1-D, e.g. to represent a waveform):

$$h(\theta, \phi) = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^l A_l^m Y_l^m(\theta, \phi)$$

Free oscillations II

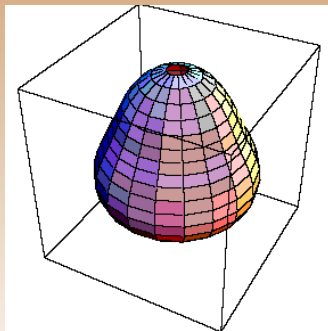
Normalized surface harmonics

- Each of the harmonics present in a particular solution describes a surface pattern with a particular pattern of symmetries.
- For normal modes, the pattern continues down into the Earth modulated by $R(r)$.
- Spherical harmonics are the usual representation for global geophysical datasets (e.g. gravity, magnetics).
- The animations below show the family of oscillations with an angular order number of $l = 3$, experienced by an homogeneous sphere.

Free oscillations II

Normalized surface harmonics

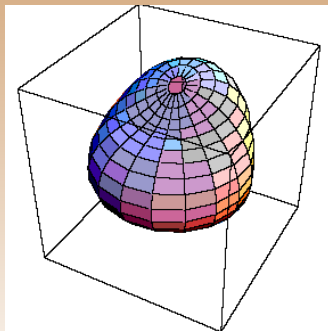
- The example below is for $l = 3, m = 0$.



Free oscillations II

Normalized surface harmonics

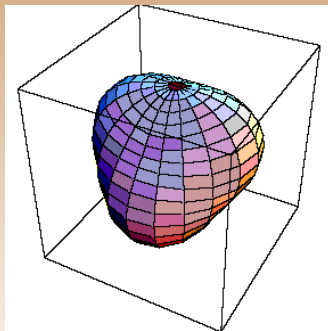
- The example below is for $l = 3, m = 1$.



Free oscillations II

Normalized surface harmonics

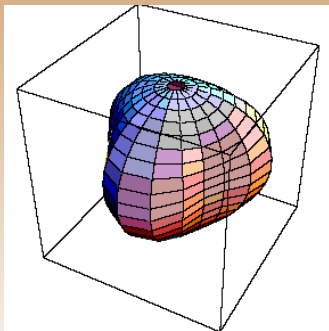
- The example below is for $l = 3, m = 2$.



Free oscillations I

Normalized surface harmonics

- The example below is for $l = 3, m = 3$.



Free oscillations II

The radial eigenfunctions

- In the previous section, we separated the (θ, ϕ) part of the solution to leave the radial equation:

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \left[\left(\frac{\omega}{c} \right)^2 - l(l+1) \right] R = 0$$

- The general solution to this class of equation is $j_l(\omega r/c)$, where $j_l(x)$ is the **spherical Bessel function** of order l .
- A general formula for this function is:

$$j_l(x) = x^l \left[\frac{-1}{x} \frac{d}{dx} \right]^l \left(\frac{\sin x}{x} \right)$$

with $x = \omega r/c$.

Free oscillations II

The radial eigenfunctions

- For $l = 0$,

$$j_0(x) = \frac{\sin x}{x}, \quad \text{so} \quad R(r) = \frac{C}{\omega r} \sin\left(\frac{C}{\omega r}\right)$$

- For $l = 1$

$$\begin{aligned} j_1(x) &= x \left(\frac{-1}{x} \right) \frac{d}{dx} \left(\frac{\sin x}{x} \right) \\ &= - \left[\frac{-\sin x}{x^2} + \frac{\cos x}{x} \right] \\ &= \frac{1}{x^2} (\sin x - x \cos x) \end{aligned}$$

Free oscillations II

The radial eigenfunctions

- Thus, we can write $R(r)$ as

$$R(r) = \left(\frac{c}{\omega r}\right)^2 \left[\sin\left(\frac{\omega r}{c}\right) - \left(\frac{\omega r}{c}\right) \cos\left(\frac{\omega r}{c}\right) \right]$$

- Boundary conditions at the surface determine permissible values of oscillation frequency ω .
- Note that the amplitude is undetermined, unless we specify the total energy in the oscillation.

Free oscillations II

Pressure modes in a homogeneous fluid sphere

- For the free oscillations of a homogeneous fluid sphere, the shear potentials are zero (a fluid does not support S-waves - $\mu = 0$.)
- Thus, we are only concerned with oscillations in dilatation θ , which are proportional to oscillations in pressure p , since $\theta = \nabla^2\Phi = p/\kappa$.
- From before, we had that (with $c = \alpha = \sqrt{\lambda/\rho}$):

$$\left(\frac{\omega}{\alpha}\right)^2 \nabla^2\Phi + \nabla^4\Phi = 0$$

so

$$\left(\frac{\omega}{\alpha}\right)^2 p + \nabla^2 p = 0$$

so pressure satisfies Helmholtz' equation.

Free oscillations II

Pressure modes in a homogeneous fluid sphere

- The external boundary condition (at $r = a$, the surface of the sphere) is zero traction $\sigma_{rr} = \lambda\theta = 0$. Since $\theta = P/\kappa$, this equates to $p = 0$ at $r = a$.
- When we solved Helmholtz' equation, we assumed that the displacement function was separable i.e.
 $f(r, \theta, \phi) = R(r)Q(\theta)F(\phi)$. In this case, when we are dealing with oscillations in pressure p , then
 $p(r, \theta, \phi) = R(r)Q(\theta)F(\phi)$.
- The boundary condition is therefore satisfied if $R(a) = 0$
- If $l = 0$, then

$$R(r = a) = \frac{c}{\omega a} \sin\left(\frac{\omega a}{c}\right) = 0$$

noting that $\omega a/c \neq 0$.

Free oscillations II

Pressure modes in a homogeneous fluid sphere

- From the above expression, it is clear that the boundary conditions can only be satisfied if:

$$\frac{\omega a}{c} = (n + 1)\pi, \quad n = 0, 1, 2, \dots$$

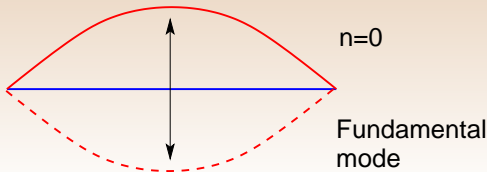
which defines a discrete spectrum of oscillations.

- The permitted frequencies are therefore given by:

$$\omega_n = \frac{c(n + 1)\pi}{a}$$

where $c = \alpha$.

- The fundamental mode occurs for $n = 0$.



Free oscillations II

Pressure modes in a homogeneous fluid sphere

- $n \geq 1$ gives overtones or higher modes.
- Nodal surfaces occur at the boundary, and we have n internal nodal surfaces where the sign of the displacement reverses.
- The above equation shows that the frequency of the oscillation increases with n .
- The eigenfunctions associated with solutions of the Helmholtz equation are labelled with the angular and radial order number l and n , as ${}_nS_l$ (spheroidal modes), or ${}_nT_l$ (torsional modes), and the frequencies associated with those eigenfunctions are labelled as ${}_n\omega_l$.
- Note that, although the frequency increases with increasing l , the azimuthal order number m does not appear in the radial equation, and therefore cannot affect the frequency of oscillation.

Free oscillations II

Pressure modes in a homogeneous fluid sphere

- Solutions are said to be **degenerate** with respect to azimuthal order number (all vibrate with the same frequency), and the degeneracy is said to be $(2l + 1)$ -fold since $-l \leq m \leq l$.
- However, very small frequency separation is observed for modes differing in m due to planetary rotation and departures from spherical symmetry.
- The complete solution for the oscillations in pressure of the fluid sphere may consist of a sum of modes:

$$p(r, \theta, \phi) = \sum_n \sum_l \sum_m \left[{}_n A_l^m j_l \left(\frac{n\omega_l r}{\alpha} \right) Y_l^m(\theta, \phi) \right]$$

where the amplitude coefficients ${}_n A_l^m$ of the individual modes are determined by the excitation function which causes the oscillations (i.e. the magnitude, location and type of earthquake).

Free oscillations II

Pressure modes in a homogeneous fluid sphere

- If the Earth was composed of a fluid with wavespeed $\alpha = 5$ km/s, and $a = 6,371$ km, then the angular frequency of the fundamental mode ($n = 0, l = 0 \Rightarrow m = 0$) is :

$${}_0\omega_0 = \frac{\alpha\pi}{a}$$

and the corresponding period is

$$T = \frac{2\pi}{{}_0\omega_0} = \frac{2\pi a}{\alpha\pi} = \frac{2a}{\alpha}$$

- Substitution of the above values give $T = 2548 \text{ s} \approx 43$ mins.
- The period of the overtones will be $43/(n+1)$ minutes for $n \geq 1$.

Free oscillations II

Pressure modes in a homogeneous fluid sphere

- The $l = 0$ mode is sometimes referred to as the “breathing” mode since angular dependence is zero. This corresponds to the planet everywhere moving in and out at the same time:

$$Y_0^0(\theta, \phi) = K_0^0 P_0^0(\cos \theta) \exp[i(m = 0)\phi] = K_0^0$$

since $P_0^0(\cos \theta) = 1$. Thus, the normalised surface harmonic is constant.

- In the Earth, the fundamental spheroidal mode ${}_0S_0$ has a period of 20 minutes (different because we neglected shear rigidity and variation of elastic parameters with depth).
- The ${}_0S_1$ mode is not physically realizable because it would correspond to the entire planet moving in one direction at frequency ω . The ${}_0S_2$ mode (football mode) is observed and has the longest period of any fundamental mode at 54 minutes.

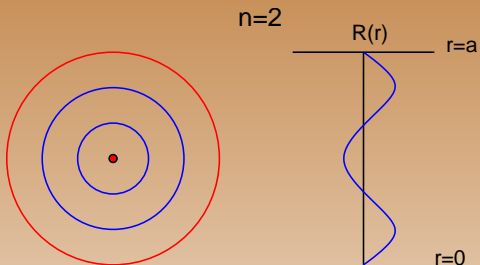
Free oscillations II

Nodal surfaces

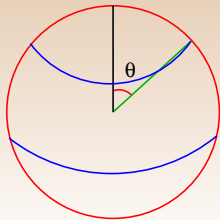
- Nodal surfaces are surfaces of zero displacement. They occur for constant radius (r), co-latitude (θ) and longitude (ϕ).
- Recall that for spheroidal or torsional modes, each 1-D part of the eigenfunction has nodes where the net displacement is zero; in 3-D, these points becomes surfaces.
- Nodes in the radial function are actually concentric spherical surfaces where displacement changes sign when the spherical Bessel function is zero. The number of spherical nodal surfaces is given by the radial order number n .
- Similarly, the nodes in the Legendre polynomial become nodal surfaces along conical sheets from $r = 0$ to $r = a$, along lines of constant co-latitude. The number of nodal surfaces in this case is given by $l - |m|$.

Free oscillations II

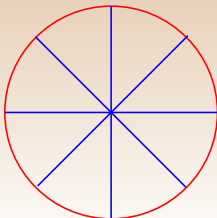
Nodal surfaces



$$l-|m|=2$$



$m=4$
view from "top"

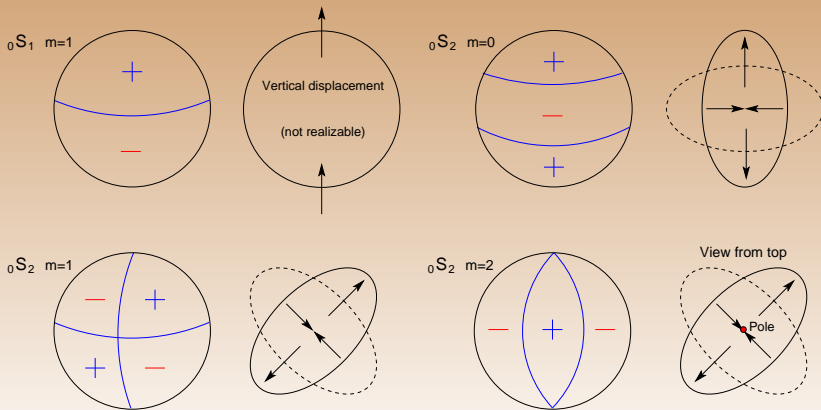


- Finally, there are $|m|$ azimuthal nodal planes containing the origin and pole. These surfaces of constant ϕ are equally spaced.

Free oscillations II

Nodal surfaces

- The schematic examples below illustrate the displacement patterns for several spheroidal modes



Free oscillations II

Torsional modes in a homogeneous elastic sphere

- From before, we had that:

$$\frac{\rho\omega^2}{r}\nabla_h^2\left(\frac{T}{r}\right) + \frac{\mu}{r}\nabla^2\nabla_h^2\left(\frac{T}{r}\right) = 0$$

which is the equation for torsional potential in an homogeneous elastic sphere.

- Multiplying both sides by r/μ and replacing $\beta = \sqrt{\mu/\rho}$ gives:

$$\left(\frac{\omega}{\beta}\right)^2\nabla_h^2\left(\frac{T}{r}\right) + \nabla^2\nabla_h^2\left(\frac{T}{r}\right) = 0$$

- This can be written:

$$\nabla_h^2\left[\left(\frac{\omega}{\beta}\right)^2\frac{T}{r} + \nabla^2\left(\frac{T}{r}\right)\right] = 0$$

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- Thus,

$$\left(\frac{\omega}{\beta}\right)^2 \frac{T}{r} + \nabla^2 \left(\frac{T}{r}\right) = f(\theta, \phi)$$

and the equation $\nabla_h^2 f = 0$ reduces to the Legendre equation with $K = l = 0$.

- The solution with $l = m = 0$ is $f(\theta, \phi) = C$, where C is a constant. This constant does not effect \mathbf{u} , so we choose $f = 0$.
- Therefore,

$$\left(\frac{\omega}{\beta}\right)^2 \frac{T}{r} + \nabla^2 \left(\frac{T}{r}\right) = 0$$

which is Helmholtz' equation for T/r .

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- The solutions for T/r are therefore exactly analogous to the solutions for the pressure oscillations of the fluid sphere, except that the phase velocity is β instead of α ; an additional factor of r appears; and the boundary condition at $r = a$ is different.
- As before, the solution is separated into parts with r , θ and ϕ dependence, so:

$$T(r, \theta, \phi) = r j_l \left(\frac{n\omega_l r}{\beta} \right) Y_l^m(\theta, \phi)$$

where j_l is the spherical Bessel function and Y_l^m is the normalised spherical harmonic.

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- For torsional mode oscillations, the relationship between displacement and potential is:

$$\begin{aligned}\mathbf{u} &= \nabla \times (T\hat{\mathbf{r}}) \\ &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r \sin \theta \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ T & 0 & 0 \end{vmatrix} \\ &= \frac{1}{r^2 \sin \theta} \left[(0)\hat{\mathbf{r}} + r \left(\frac{\partial T}{\partial \theta} \right) \hat{\boldsymbol{\theta}} - r \sin \theta \left(\frac{\partial T}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \right] \\ &= \left[0, \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi}, -\frac{1}{r} \frac{\partial T}{\partial \theta} \right]\end{aligned}$$

- Thus, the torsional displacement has no radial component.

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- At the free surface ($r = a$), we can impose the zero traction boundary condition. This requires that $\sigma_{r\phi} = 0$.
- Using our stress strain relationship $\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\mathbf{e}_{ij}$, then $\sigma_{r\phi} = 2\mu\mathbf{e}_{ij} = 0$ at $r = a$.
- The components of the symmetric strain tensor can be written:

$$\mathbf{e}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

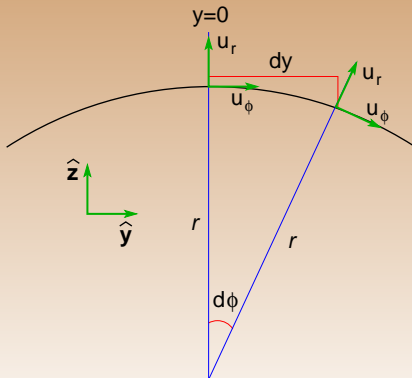
- However, the above Cartesian expression for strain must be expressed in terms of the spherical components of displacement:

$$e_{r\theta} = \frac{1}{2} \left[\frac{\partial}{\partial z} (\mathbf{u} \cdot \hat{\mathbf{y}}) + \frac{\partial}{\partial y} (\mathbf{u} \cdot \hat{\mathbf{z}}) \right] = 0$$

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- In the above expression, $\hat{\mathbf{y}}$ is a unit vector in the ϕ direction, and $\hat{\mathbf{z}}$ is a unit vector in the r direction.



- At $y = 0$, $\mathbf{u} \cdot \hat{\mathbf{z}} = u_r$, so there is no ϕ component of displacement.
- At $y = dy$, we must consider both u_r and u_ϕ , since they both have z -components of displacement.

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- If u_{zr} represents the z component of u_r , then $\cos d\phi = u_{zr}/u_r$.
- Similarly, if $u_{z\phi}$ represents the z component of u_ϕ , then $\sin d\phi = u_{z\phi}/u_\phi$
- Thus,

$$\begin{aligned}\mathbf{u} \cdot \hat{\mathbf{z}} &= u_{zr} + u_{z\phi} = u_r \cos d\phi - u_\phi \sin d\phi \\ &= u_r - u_\phi d\phi\end{aligned}$$

since in general, $\cos(x \rightarrow 0) = 1$ and $\sin(x \rightarrow 0) = x$.

- Therefore,
$$d\mathbf{u} \cdot \hat{\mathbf{z}} = d\mathbf{u} \cdot \hat{\mathbf{z}}(y = dy) - d\mathbf{u} \cdot \hat{\mathbf{z}}(y = 0) = u_r - u_\phi d\phi - u_r.$$

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- Since $dy = r \sin d\phi = rd\phi$,

$$\frac{\partial \mathbf{u} \cdot \hat{\mathbf{z}}}{\partial y} = -\frac{u_\phi d\phi}{rd\phi} = \frac{-u_\phi}{r}$$

- In the radial direction, $\partial u_y / \partial z = \partial u_\phi / \partial r$, so

$$\frac{\partial \mathbf{u} \cdot \hat{\mathbf{z}}}{\partial y} = \frac{-u_\phi}{r} \quad \text{and} \quad \frac{\partial \mathbf{u} \cdot \hat{\mathbf{y}}}{\partial z} = \frac{\partial u_\phi}{\partial r}$$

- Therefore, the boundary condition is:

$$\sigma_{r\phi} = \mu \left[\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] = 0$$

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- Using our expression for the ϕ component of displacement derived earlier,

$$\frac{\partial}{\partial r} \left(\frac{-1}{r} \frac{\partial T}{\partial \theta} \right) - \frac{1}{r} \left(\frac{-1}{r} \right) \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial r} \left(\frac{-T}{r} \right) + \frac{1}{r^2} T \right] = 0$$

$$\frac{\partial}{\partial \theta} \left[\frac{1}{r} \frac{\partial T}{\partial r} + \frac{2T}{r^2} \right] = 0$$

- The term in brackets must also be equal to zero because it is invariant to changes in θ :

$$\frac{1}{r} \frac{\partial T}{\partial r} + \frac{2T}{r^2} = 0$$

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- We will now apply this boundary condition to the radial part of the eigenfunction with $l = 1$ (i.e. the spherical Bessel function $j_1(x)$).
- From before, this eigenfunction has the form:

$$T(r) = r \left(\frac{\beta}{r\omega} \right)^2 \left[\sin \left(\frac{r\omega}{\beta} \right) - \left(\frac{r\omega}{\beta} \right) \cos \left(\frac{r\omega}{\beta} \right) \right]$$

- The boundary condition can be written:

$$\frac{1}{r} \frac{\partial T}{\partial r} + \frac{2T}{r^2} = \left[\frac{d}{dr} - \frac{1}{r} \right] \frac{T}{r} = 0$$

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- Substitution of T into the above equation gives the following equation at $r = a$:

$$\left[\frac{d}{dr} - \frac{1}{r} \right] \left\{ \left(\frac{\beta}{r\omega} \right)^2 \left[\sin \left(\frac{r\omega}{\beta} \right) - \left(\frac{r\omega}{\beta} \right) \cos \left(\frac{r\omega}{\beta} \right) \right] \right\} = 0$$

- Simplifying this expression and substituting $r = a$ gives:

$$\left[1 - 3 \left(\frac{\beta}{a\omega} \right)^2 \right] \sin \left(\frac{a\omega}{\beta} \right) + 3 \left(\frac{\beta}{a\omega} \right) \cos \left(\frac{a\omega}{\beta} \right) = 0$$

- This can be written:

$$\left[1 - \frac{3}{x^2} \right] \sin x + \frac{3}{x} \cos x = 0$$

where $x = a\omega/\beta$.

Free oscillations II

Torsional modes in a homogeneous elastic sphere

- Dividing through by $\cos x$ yields:

$$\tan x = \frac{3x}{3 - x^2}$$

- This equation can be numerically solved for x . It turns out to have an infinite number of solutions, which can be seen by plotting the left hand side against the right hand side.
- The fundamental mode ${}_0T_0$ does not occur as it would correspond to oscillatory changes in the total angular momentum of the Earth.
- ${}_0T_1$ has a period of approximately 25 minutes.

Free oscillations II

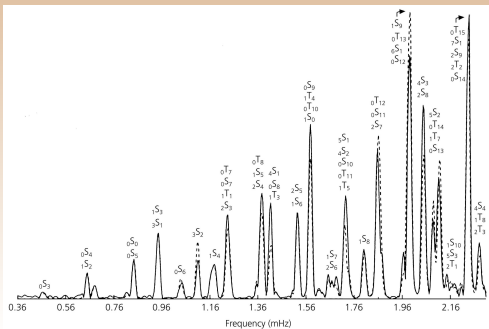
Eigenfunctions for a spherically symmetric Earth

- In the above treatment, the Earth was considered to be an homogeneous medium, which is not a very good approximation.
- However, to a good approximation, the Earth is spherically symmetric - $\lambda(r)$, $\mu(r)$ and $\rho(r)$.
- In this case, we get a modified form of the Helmholtz equations governing torsional and spheroidal modes.
- Like the homogeneous case, the equations are separable into (r, θ, ϕ) dependence of solution. However, the radial equation has numerical solutions only.
- The frequency of oscillation is still determined by the radial eigenfunctions and appropriate boundary conditions. Degeneracy with respect to the m -values still occurs.

Free oscillations II

Observations

- Normal modes can be observed as very low frequency spectral peaks in the frequency domain of the recorded waveform following large earthquakes.
- The frequency of many specific modes have been accurately determined. They can be used to constrain depth variation of density and elastic parameters by comparison with theoretical predictions.



- Amplitude spectrum of the radial component of a 35-hour seismogram recorded from the June 9, 1994 Bolivia earthquake.