PEAT8002 - SEISMOLOGY
Lecture 2: Continuum mechanics

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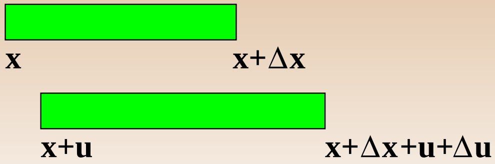
Strain

- Strain is the formal description of the change in shape of a material.
- We want to be able to describe the deformation of a continuum; to do this, we can use the displacement vector $\mathbf{u}(\mathbf{x}, t) = \mathbf{x}(t) - \mathbf{x}(t_0)$, which describes the location of every point relative to its position at a reference time t_0.
- Consider an element of solid material within which displacements $\mathbf{u}(\mathbf{x})$ have occurred.



- What is the point $\mathbf{x} + \Delta\mathbf{x}$ displaced by?

This can be done using a Taylor series expansion:

$$\mathbf{u}(\mathbf{x}+\Delta\mathbf{x}) = \mathbf{u}(\mathbf{x}) + J\Delta\mathbf{x} + O(\Delta\mathbf{x}^2)$$

where

$$J = \begin{bmatrix} \partial u_x/\partial x & \partial u_x/\partial y & \partial u_x/\partial z \\ \partial u_y/\partial x & \partial u_y/\partial y & \partial u_y/\partial z \\ \partial u_z/\partial x & \partial u_z/\partial y & \partial u_z/\partial z \end{bmatrix}$$

J is often described as the **Deformation gradient tensor.** If we ignore the second and higher order terms in the expansion, then

$$\Delta\mathbf{u} = J\Delta\mathbf{x}$$

- In seismological applications, we ignore higher order deformation terms; usually, Earth strains are small enough to validate this approximation.
- Typical strain values due to the passage of seismic waves are $10^{-3} \Rightarrow 10^{-4}$. For far field waves, strain is often $< 10^{-6}$.
- The deformation gradient tensor describes both rigid body rotation and deformation; we are interested in the latter. It is possible to separate these two effects by dividing J into symmetric and anti-symmetric parts.

$$J = e + \Omega = \nabla \mathbf{u} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + \frac{1}{2}[\nabla \mathbf{u} - (\nabla \mathbf{u})^T]$$

- The symmetric **strain tensor** ($e_{ij} = e_{ji}$) is given by

$$\mathbf{e} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

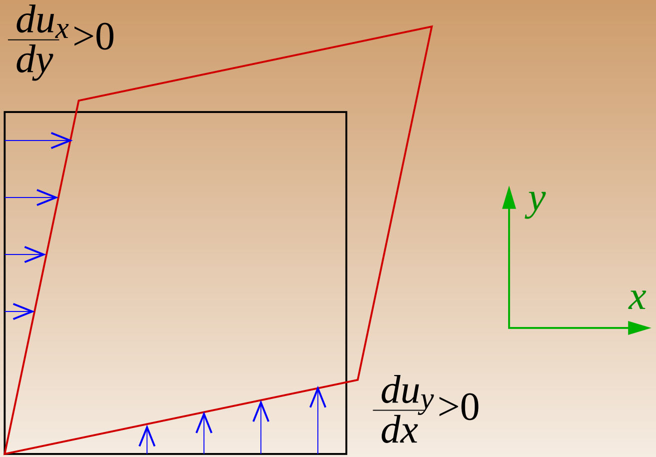
- The anti-symmetric **rotation tensor** Ω ($\Omega_{ij} = -\Omega_{ji}$) is given by

$$\Omega = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) & 0 \end{bmatrix}$$

If $\Omega = 0$ and there is no volume change ($\partial u_x/\partial x = \partial u_y/\partial y = \partial u_z/\partial z = 0$), then if we consider a 2-D square

$$J = \mathbf{e} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & 0 \end{bmatrix}$$

with $\partial u_x/\partial y = \partial u_y/\partial x$ since $\Omega = 0$. The deformation of the square will therefore look as follows (bearing in mind the first-order approximation) if $\partial u_x/\partial y > 0$ (which $\Rightarrow \partial u_y/\partial x > 0$):

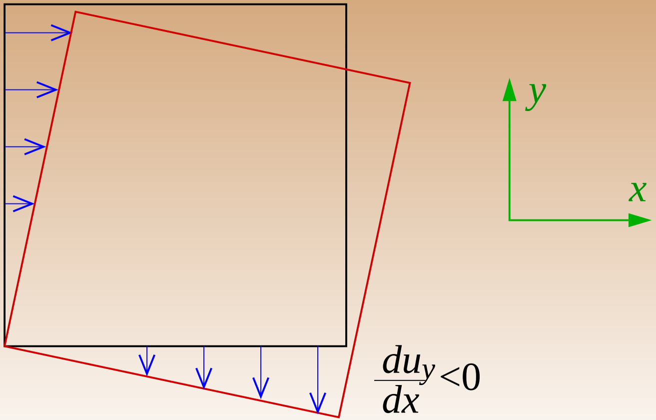

$\frac{du_x}{dy} > 0$
$\frac{du_y}{dx} > 0$
y
x

If $\mathbf{e} = \mathbf{0}$, then $\partial u_x / \partial y = -\partial u_y / \partial x$ and

$$J = \Omega = \begin{bmatrix} 0 & \dfrac{\partial u_x}{\partial y} \\ \dfrac{\partial u_y}{\partial x} & 0 \end{bmatrix}$$

The deformation of the square is now purely rotational, and will look as follows if $\partial u_x / \partial y > 0$ (which $\Rightarrow \partial u_y / \partial x < 0$):

$$\frac{du_x}{dy} > 0$$



$$\frac{du_y}{dx} < 0$$

- Often, the strain tensor is expressed using index notation.

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- The change in volume of a material, or dilatation, is given by:

$$\theta = e_{xx} + e_{yy} + e_{zz} = e_{ii} = \mathrm{tr}[\mathbf{e}] = \nabla \cdot \mathbf{u}$$

- Einstein summation convention: when an index appears twice in a single term, it implies summation over that index e.g.

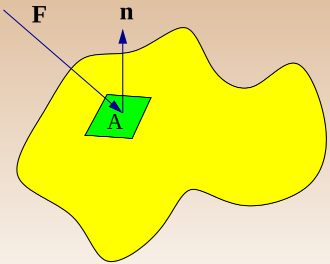
$$w_{ijk} a_i b_j c_k = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} w_{ijk} a_i b_j c_k$$

Stress

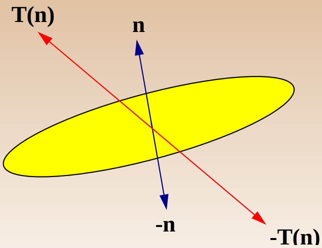
- Non-zero strain of an elastic material requires some internal force distribution, which can be described by the stress field.
- Like strain, stress is a tensor quantity.
- Forces acting on elements of the continuum are of two types:
    - Body forces **f** (e.g. gravity)
    - Surface tractions **T** exerted by neighbouring elements of the continuum or by external forces applied to the boundary of the continuum.

- Traction is the force per unit area acting on a surface within the continuum and can be defined by:

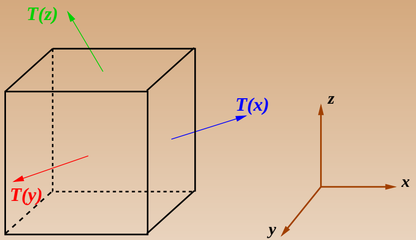
$$\mathbf{T}(\hat{\mathbf{n}}) = \lim_{A \to 0} \left[\frac{\mathbf{F}}{A(\hat{\mathbf{n}})} \right]$$



- Traction has the same orientation as the applied force, but is a function of the orientation of the surface defined by the unit normal vector $\hat{\mathbf{n}}$.
- Traction is symmetric with respect to reflection in the surface, i.e. $\mathbf{T}(-\hat{\mathbf{n}}) = -\mathbf{T}(\hat{\mathbf{n}})$.



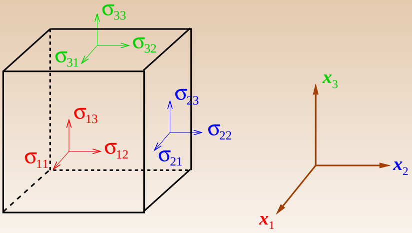
- We need three orthogonal surfaces to describe a complete state of force at any point.



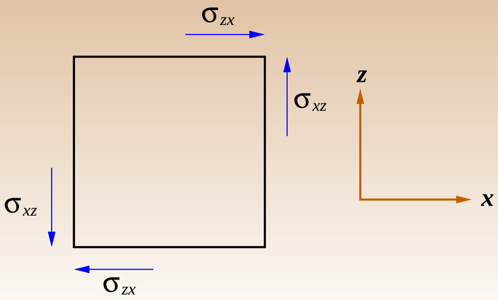
- where:
  - $\hat{\mathbf{x}}$ = unit normal vector in direction of x axis.
  - $\hat{\mathbf{y}}$ = unit normal vector in direction of y axis.
  - $\hat{\mathbf{z}}$ = unit normal vector in direction of z axis.

- The complete stress state is described by three 3-component vectors ⇒ tensor.
- The stress tensor is therefore defined by:

$$\boldsymbol{\sigma} = \begin{bmatrix} \mathbf{T}(\hat{\mathbf{x}}) \\ \mathbf{T}(\hat{\mathbf{y}}) \\ \mathbf{T}(\hat{\mathbf{z}}) \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



- Due to the conservation of angular momentum, there can be no net rotation from shear stresses.
- Therefore $\sigma_{ij} = \sigma_{ji}$ and the stress tensor is symmetric and contains only six independent elements. These are sufficient to completely describe the state of stress at a given point in the medium.



- The stress tensor therefore may be written as:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

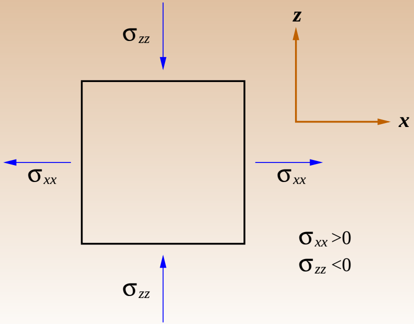
- Normal and shear stress components are simply defined by the diagonal and off-diagonal elements of $\boldsymbol{\sigma}$:

$$\sigma_{ij} = \begin{cases} \text{normal stress if } i = j \text{ (normal to plane)} \\ \text{shear stress if } i \neq j \text{ (parallel to plane)} \end{cases}$$

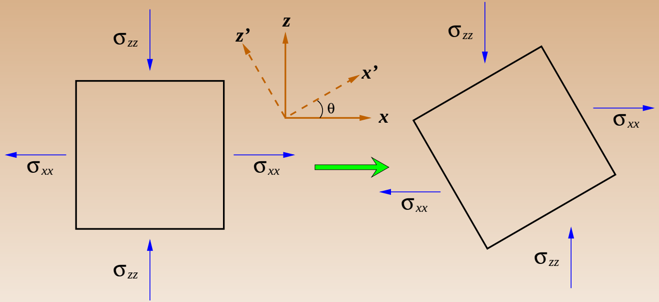
- Consider a 2-D box with faces aligned to the Cartesian coordinate axes (x, z) subject to a stress field defined by:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{zz} \end{bmatrix}$$

- The box is therefore subject only to normal stress.



- Consider a second box that is now oriented at an angle θ to the original box but is subject to the same stress field. What are the traction components that act on the side of the second box?



- This problem can be solved by rotating the coordinate axes to align them with the new box.

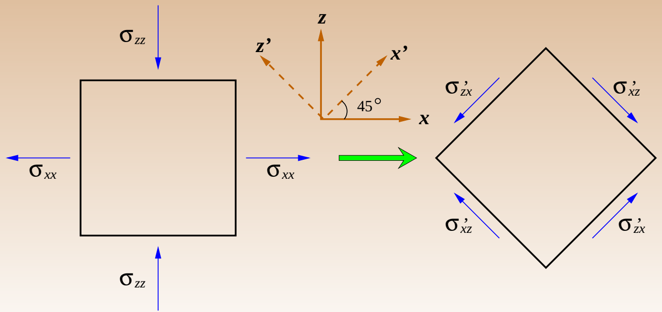
- In order to specify the stress tensor in the rotated coordinate system (x', z'), we make use of the relationship $\boldsymbol{\sigma}' = A\boldsymbol{\sigma}A^{\mathsf{T}}$, where A is the Cartesian transformation matrix.
- This is the same matrix used to transform a vector **u** from one coordinate system to another (i.e. $\mathbf{u}' = A\mathbf{u}$).

$$\begin{aligned}\boldsymbol{\sigma}' &= \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sigma_{xx} & 0 \\ 0 & \sigma_{zz}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \\ &= \begin{bmatrix}\sigma_{xx}\cos^2\theta + \sigma_{zz}\sin^2\theta & (\sigma_{zz} - \sigma_{xx})\cos\theta\sin\theta \\ (\sigma_{zz} - \sigma_{xx})\cos\theta\sin\theta & \sigma_{xx}\sin^2\theta + \sigma_{zz}\cos^2\theta\end{bmatrix}\end{aligned}$$

- If we set $\theta = 45^\circ$ and $\sigma_{xx} = -\sigma_{zz}$, then:

$$\boldsymbol{\sigma}' = \begin{bmatrix} 0 & \sigma_{zz} \\ \sigma_{zz} & 0 \end{bmatrix}$$

and only shear stresses act on the smaller block.



- For an arbitrary stress field, the traction vector will in most cases have non-zero components parallel and perpendicular to any surface orientation within the continuum.
- However, for any state of stress, surfaces can always be orientated in such a way that the shear tractions vanish.
- The normal vectors to these surfaces are oriented parallel to the principal stress axes, and the normal stresses acting on the surfaces are called the principal stresses.
- In order for the shear tractions to vanish for a surface defined with the unit normal vector $\hat{\mathbf{n}}$, then the total traction acting on the surface must be parallel to $\hat{\mathbf{n}}$. Thus, $\mathbf{T}(\hat{\mathbf{n}}) = \lambda\hat{\mathbf{n}}$.

- The traction across any arbitrary plane of orientation $\hat{\mathbf{n}}$ can be obtained by:

$$\mathbf{T}(\hat{\mathbf{n}}) = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix}$$

- Since we have that $\lambda\hat{\mathbf{n}} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, then

$$\sigma \cdot \hat{\mathbf{n}} - \lambda\hat{\mathbf{n}} = 0$$
$$(\sigma - \mathrm{I}\lambda)\hat{\mathbf{n}} = 0$$

- which only has a nontrivial solution when

$$\det[\sigma - \mathrm{I}\lambda] = 0$$

- Thus, the principal stress axes $\hat{\mathbf{n}}$ are the eigenvectors of the tensor, and associated principal stresses λ are the eigenvalues.
- Computing the determinant yields the characteristic polynomial:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

- The roots of the above cubic polynomial define the three principal stresses, and substitution of each of these into $(\sigma - \mathrm{I}\lambda)\hat{\mathbf{n}} = 0$ yields the associated eigenvector, which corresponds to a principal stress direction.

- It turns out that it is always possible to find three orthogonal normal vectors $\hat{\mathbf{n}}$. The transformation matrix is therefore $A = [\hat{\mathbf{n}}^{(1)}, \hat{\mathbf{n}}^{(2)}, \hat{\mathbf{n}}^{(3)}]^{\mathrm{T}}$.
- The stress tensor in the new coordinate system $\boldsymbol{\sigma}' = A\boldsymbol{\sigma}A^{\mathrm{T}}$ is diagonal, and is commonly written as:

$$\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

- The usual convention is that σ_1 is in the direction of the maximum principal stress, σ_2 the intermediate principal stress, and σ_3 the minimum principal stress.

- The stress field within the Earth is usually dominated by the compressive stresses caused by the weight of overlying rock.
- In many applications, only the deviations from this dominant compressive stress are of interest.
- The **Deviatoric stress** τ is defined as

$$\tau_{ij} = \sigma_{ij} - M\delta_{ij}$$

- In the above definition, M is the mean stress, which is defined as the average normal stress:

$$M = \frac{\sigma_{kk}}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

- It turns out that the trace of the stress tensor is invariant under rotation of the coordinate system.
- Therefore, the mean stress can be written in terms of the principal stress since $\sigma'_{kk} = \sigma_{ii}$:

$$M = \frac{\sigma'_{kk}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

- When the principal stresses are large and nearly equal, the deviatoric stress tensor removes their effect and indicates the remaining stress state.

- In a fluid at rest (or in which viscosity is negligible), the traction is always parallel to $\hat{\mathbf{n}}$ (i.e. fluid does not support shear stresses) and its magnitude is independent of orientation.
- In this case, traction is simply equal to $p\hat{\mathbf{n}}$ where $p(\mathbf{x}, t)$ is the **hydrostatic pressure**. Therefore:

$$\boldsymbol{\sigma} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

- When $\sigma_{ij} = p\delta_{ij}$ (equivalent to above expression), the stress field is referred to as **isotropic**
- Note that:

$$\delta_{ij} = \left\{ \begin{array}{lll} 1 & \text{for} & i = j \\ 0 & \text{for} & i \neq j \end{array} \right\} = \text{Kronecker} \quad \text{delta}$$

- An elastic medium is one in which strain is a linear function of the applied stress.
- Since both stress and strain are 3×3 tensor quantities, the most general possible elastic medium has a constitutive equation of the form:

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} e_{kl} = C_{ijkl} e_{kl}$$

- **C**, or the elastic tensor, is a 4th order tensor $(3 \times 3 \times 3 \times 3)$ and could conceivably have $3^4 = 81$ independent elements.

- Symmetry of stress and strain tensors requires that:

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$

  which reduces the number of independent components to 36 (six independent components of stress and strain).
- A final symmetry, $C_{ijkl} = C_{klij}$, can be derived from thermodynamics to prove that there are, at most, 21 independent elastic constants.
- The constants C_{ijkl} are often referred to as the elastic moduli, and describe the elastic properties of the medium.

- In an isotropic medium, the elastic properties are independent of orientation; the response (strain) is the same for stress applied in any direction.
- The general elastic tensor then reduces to a simpler form, depending on only two constants, which may be expressed in various ways.
- One useful pair are the Lamé constants λ and μ, which are defined such that

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Each component of the stress field can be written in terms of strain as follows (noting the implied summation)

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}e_{kl} \\ &= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]e_{kl} \\ &= \lambda\delta_{ij}(e_{xx} + e_{yy} + e_{zz}) + \mu\delta_{ik}\delta_{jl}e_{kl} + \mu\delta_{il}\delta_{jk}e_{kl}\end{aligned}$$

Each of the six independent components of the stress field can then be explicitly written

$$\begin{aligned}\sigma_{xx} &= \lambda\theta + 2\mu e_{xx} \\ \sigma_{yy} &= \lambda\theta + 2\mu e_{yy} \\ \sigma_{zz} &= \lambda\theta + 2\mu e_{zz} \\ \sigma_{xy} &= \mu(e_{xy} + e_{yx}) = 2\mu e_{xy} \\ \sigma_{xz} &= 2\mu e_{xz} \\ \sigma_{yz} &= 2\mu e_{yz}\end{aligned}$$

- These expressions can be summarised as Hooke's law for an isotropic solid

$$\sigma_{ij} = \lambda\delta_{ij}\theta + 2\mu e_{ij}$$

- Thus, the two Lamé parameters λ and μ completely describe the linear stress-strain relationship within an isotropic solid.
- Validity: isotropic solid; short time scale; small deformation, perfect elasticity.
- Isotropy is a reasonable approximation for much of the Earth's interior.

- Elastic “constants” are supposedly independent of strain, but typically depend on temperature, pressure, and composition, and so vary with respect to location within the Earth.
- The assumption of perfect elasticity does not allow for the observed dissipation of seismic wave energy in the course of propagation.
- On long time scales and for large strains, anelasticity is important, requiring a different constitutive relationship, e.g. in a viscous material, we prescribe a relationship between stress and strain rate $\partial \mathbf{e}/\partial t$.

- In the above equations, μ is referred to as the **shear modulus** and can be defined as half the ratio between the applied shear stress and the resulting shear strain

$$\mu = \frac{\sigma_{xz}}{2e_{xz}}$$

When μ is large, then the shear stress is large compared to the resulting shear strain, which implies rigidity. When μ is small, then the shear stress is small compared to the resulting shear strain, which implies low rigidity. For a fluid, $\mu = 0$.

- The other Lamé parameter λ does not have a simple physical explanation

$$\lambda = \frac{\sigma_{xx} - 2\mu e_{xx}}{\theta} = \frac{\sigma_{xx} - \sigma_{xz} e_{xx} / e_{xz}}{e_{xx} + e_{yy} + e_{zz}}$$

- **Young's modulus E:** This is defined as the ratio of extensional stress to the resulting extensional strain for a cylinder being pulled apart at both ends

$$E = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu}$$

- **Bulk modulus κ:** This is defined as the ratio of hydrostatic pressure to the resulting volume change - it is a measure of the incompressibility of the material

$$\kappa = \frac{p}{\theta} = \lambda + \frac{2}{3}\mu$$

where $p = \frac{1}{3}\sigma_{kk}$.

- **Poisson's ratio ν:** Poisson's ratio is defined as the ratio of the lateral contraction of a cylinder (pulled on its ends) to its longitudinal extension

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

- E, ν and κ are often used in engineering because they are easily measured by simple experiments. In seismic wave propagation, λ, μ and κ are more natural constants.