PEAT8002 - SEISMOLOGY
Lecture 3: The elastic wave equation

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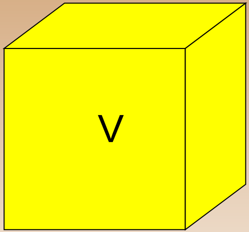
- The equation of motion can be derived by considering the total force applied to a volume element V with surface area S.



- We require the sum of the surface force field, derived from the tractions $\mathbf{T}$, and the external forces to balance the mass acceleration in the volume, i.e. $\mathbf{F} = m\mathbf{a}$.

- The force balance equation can be written as:

$$\mathbf{F}_{\text{SURFACE}} + \mathbf{F}_{\text{BODY}} = \mathbf{F}_{\text{TOTAL}}$$

$$\mathbf{T}(\hat{\mathbf{n}}) = \lim_{A\to 0}\left[\frac{\mathbf{F}_S}{A(\hat{\mathbf{n}})}\right]$$

$$\mathbf{F}_S = \int_S \mathbf{T}(\hat{\mathbf{n}})\mathrm{d}S$$

$$\mathbf{F}_B = \int_V \rho\mathbf{g}\mathrm{d}V$$

- The bottom integral for $\mathbf{F}_B$ occurs because $\mathbf{F} = m\mathbf{g}$ (where $\mathbf{g}$ is the acceleration due to gravity) and the density is $\rho = m/V$, so $\mathrm{d}\mathbf{F}_B = \rho\mathbf{g}\mathrm{d}V$.

- The total force, where **f** is the local acceleration field, is then given by

$$\mathbf{F}_T = \int_V \rho \mathbf{f} \mathrm{d}V$$

- Therefore, the complete force balance equation is:

$$\int_S \mathbf{T}(\hat{\mathbf{n}}) \mathrm{d}S + \int_V \rho \mathbf{g} \mathrm{d}V = \int_V \rho \mathbf{f} \mathrm{d}V$$

- Since $\mathbf{f} = \partial^2 \mathbf{u}/\partial t^2$ and $\mathbf{T}(\hat{\mathbf{n}}) = \hat{n}_i \mathbf{T}(x_i) = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$,

$$\int_S \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \mathrm{d}S + \int_V \rho \mathbf{g} \mathrm{d}V = \int_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \mathrm{d}V$$

- Now, according to the **divergence theorem** (transformation between volume integrals and surface integrals),

$$\int_S \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \mathrm{d}S = \int_V \nabla \cdot \boldsymbol{\sigma} \mathrm{d}V$$

- Therefore we can write:

$$\int_V \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \mathrm{d}V = \int_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \mathrm{d}V$$

$$\int_V \left[\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right] \mathrm{d}V = 0$$

- The integrand must be zero since this relationship holds for any arbitrary control volume, so

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

- In the absence of body forces (not valid for very low frequencies e.g. normal modes) and away from the source,

$$\nabla \cdot \boldsymbol{\sigma} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

which is the **equation of motion**.

- We can use our expression for the equation of motion and the relationship between stress and strain in an isotropic elastic medium to derive the elastic wave equation.
- In other words, we can substitute:

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

into

$$\nabla \cdot \boldsymbol{\sigma} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

- For fully anisotropic media, we would need to substitute the expression $\sigma_{ij} = C_{ijkl} e_{kl}$ into the equation of motion, where $\{C_{ijkl}\}$ represent the 21 independent elastic moduli.

- Substitution of the isotropic stress-strain relationship into the equation of motion yields:

$$[\nabla \cdot \sigma]_i = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}[\lambda \theta \delta_{ij} + 2\mu e_{ij}]$$

- Taking the first term on the right hand side,

$$\frac{\partial}{\partial x_j}[\lambda \theta \delta_{ij}] = \frac{\partial}{\partial x_i}[\lambda \theta]$$

  which occurs because $\delta_{ij} = 1$ only if $j = i$.
- Application of the product rule to the second term yields:

$$\frac{\partial}{\partial x_j}[2\mu e_{ij}] = 2\mu \frac{\partial e_{ij}}{\partial x_j} + 2e_{ij}\frac{\partial \mu}{\partial x_j}$$

- From the previous lecture, we showed that strain is related to displacement by

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

- Putting these results together produces

$$[\nabla \cdot \sigma]_i = \frac{\partial}{\partial x_i}[\lambda\theta] + \mu\frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + 2e_{ij}\frac{\partial \mu}{\partial x_j}$$

- Each of the four summed terms on the RHS of this equation can be reorganised as follows:

- **Term 1:**

$$\left\{\frac{\partial}{\partial x_i}[\lambda\theta]\right\} = \nabla(\lambda\theta)$$

- **Term 2:**

$$\left\{\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right\} = \frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_j} = \begin{bmatrix} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \end{bmatrix}^T = \nabla^2 \mathbf{u}$$

- **Term 3:** it is assumed that **u** has continuous second partial derivatives.

$$\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)\right\} = \left\{\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)\right\}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) \end{bmatrix}$$

$$\frac{\partial}{\partial \mathbf{x}}(\nabla\cdot\mathbf{u}) = \left[\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{u})\right] = \nabla(\nabla\cdot\mathbf{u}) = \nabla\theta$$

- Note that the last equality occurs because:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \theta$$

- **Term 4:**

$$\left\{ e_{ij} \frac{\partial \mu}{\partial x_j} \right\} = \mathbf{e} \cdot \nabla \mu$$

- Putting all four terms together yields the **elastic wave equation**:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla(\lambda\theta) + \mu \nabla^2 \mathbf{u} + \mu \nabla \theta + 2\mathbf{e} \cdot \nabla \mu = \rho \ddot{\mathbf{u}}$$

- The elastic wave equation can also be written completely in terms of displacement:

$$\rho\ddot{\mathbf{u}} = \nabla\lambda(\nabla\cdot\mathbf{u})+\nabla\mu\cdot[\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathsf{T}}]+(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\times\nabla\times\mathbf{u}$$

- It applies to small displacements from an equilibrium state in an elastic medium.
- The first two terms on the RHS contain gradients of the Lamé parameters. They are non-zero if the material is inhomogeneous (i.e. contains velocity gradients).

- It can be shown that the strength of $\nabla\lambda$ and $\nabla\mu$ tend to zero when the wave frequency is high compared to the scale length of the variation in λ and μ. This approximation is used in most ray theoretical methods.
- Thus, if we ignore the gradient terms, we get:

$$\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u}$$

  which is strictly valid for homogeneous isotropic media.
- Using the above equation, we can show that elastic media support both compressional (P) and shear (S) waves.

- Rather than directly solve the wave equation derived on the previous slide, we can express the displacement field in terms of two other functions, a scalar $\Phi(\mathbf{x}, t)$ and a vector $\mathbf{\Psi}(\mathbf{x}, t)$, via Helmholtz' theorem

$$\mathbf{u} = \nabla\Phi + \nabla \times \mathbf{\Psi}$$

- In this representation, the displacement is the sum of the gradient of a scalar potential and the curl of a vector potential.
- Although this representation of the displacement field would at first appear to introduce complexity, it actually clarifies the problem because of the following two vector identities

$$\nabla \times (\nabla\Phi) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{\Psi}) = 0$$

- Due to the form of the RHS of the wave equation, these two vector identities separates the displacement field into two parts:
  - $\nabla\Phi$: No curl or rotation and gives rise to compressional waves.
  - $\nabla \times \mathbf{\Psi}$: Zero divergence; causes no volume change and corresponds to shear waves.
- If we now substitute $\mathbf{u} = \nabla\Phi + \nabla \times \mathbf{\Psi}$ into

$$\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u}$$

we obtain:

$$\rho\frac{\partial^2}{\partial t^2}(\nabla\Phi + \nabla \times \mathbf{\Psi}) = (\lambda + 2\mu)\nabla(\nabla^2\Phi) - \mu\nabla \times \nabla \times (\nabla \times \mathbf{\Psi})$$

- The right hand side of the above equation comes from the fact that if we take the divergence of the displacement field then:
$$\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \Phi + \nabla \cdot \nabla \times \boldsymbol{\Psi} = \nabla^2 \Phi$$
- Likewise, if we take the curl of the displacement field, then

$$\begin{aligned} \nabla \times \mathbf{u} &= \nabla \times \nabla \Phi + \nabla \times \nabla \times \boldsymbol{\Psi} \\ &= \nabla(\nabla \cdot \boldsymbol{\Psi}) - \nabla^2 \boldsymbol{\Psi} \\ &= -\nabla^2 \boldsymbol{\Psi} \end{aligned}$$

noting the use of the vector identity $\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times \nabla \times \mathbf{v}$ which is valid for any vector **v**.

- It is also assumed without loss of generality that $\nabla \cdot \boldsymbol{\Psi} = 0$ - the vector potential has zero divergence. This can be done since taking the curl discards any part of the vector potential that would give rise to a non-zero divergence.
- Using the same vector identity as before, the second term on the RHS of the wave equation can be simplified as follows:

$$\nabla\times\nabla\times(\nabla\times\boldsymbol{\Psi}) = -\nabla^2(\nabla\times\boldsymbol{\Psi})+\nabla(\nabla\cdot(\nabla\times\boldsymbol{\Psi}) = -\nabla^2(\nabla\times\boldsymbol{\Psi})$$

since the divergence of the curl is zero.

- This now allows the wave equation to be re-organised as follows:

$$\nabla\left[(\lambda+2\mu)\nabla^2\Phi-\rho\frac{\partial^2\Phi}{\partial t^2}\right]=-\nabla\times\left[\mu\nabla^2\mathbf{\Psi}-\rho\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right]$$

- One solution to the above equation can be obtained by setting both bracketed terms to zero. This yields two wave equations, one for each potential.

- In this case, the scalar potential satisfies:

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{(\lambda + 2\mu)}{\rho} \nabla^2 \Phi$$

which clearly has the form of a scalar wave equation i.e.

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

- The speed α at which this type of wave propagates is therefore given by:

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

- Similarly, the the vector potential satisfies:

$$\frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \boldsymbol{\Psi}$$

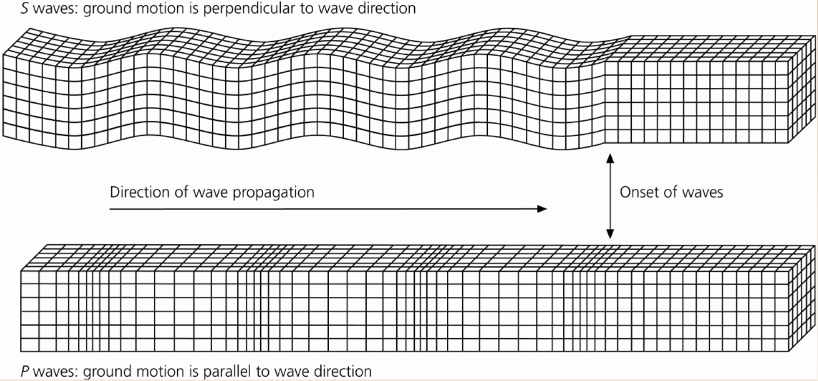
which clearly has the form of a vector wave equation i.e.

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c^2 \nabla^2 \mathbf{v}$$

- The speed β at which this type of wave propagates is therefore given by:

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

- The oscillatory variable ($\mathbf{u} = \nabla\phi$) of P-waves or compressional waves is volumetric change, which is directly related to pressure.
- In contrast, the oscillatory variable $\mathbf{u} = \nabla \times \boldsymbol{\Psi}$ of S-waves implies a shear disturbance with no volume change ($\nabla \cdot \nabla \times \boldsymbol{\Psi} = 0$).
- Although the above derivation is strictly valid for high frequency waves in isotropic media, observational studies confirm that P-waves invariably travel faster than S-waves and hence always correspond to the first-arrivals on a seismogram.



S waves: ground motion is perpendicular to wave direction
Direction of wave propagation
Onset of waves
P waves: ground motion is parallel to wave direction

- Another interesting property of these waves that is confirmed by observation is that P-waves can propagate through a liquid but S-waves cannot. This is because liquids do not support a shear stress ($\mu = 0$), so $\alpha = \sqrt{\lambda/\rho}$ and $\beta = 0$.
- Primary seismological evidence for the existence of a liquid outer core is based on the differing behaviour of P- and S-waves.
- Note that in realistic media (i.e. the Earth), P- and S-wave motions do not completely decouple in the way described above, but in many applications, this approximation is acceptable.