PEAT8002 - SEISMOLOGY
Lecture 4: Body waves

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- P- and S-waves are referred to as body waves because they are transmitted through the volume (or body) of the material, as apposed to travelling along the surface.
- Both types of body waves are non-dispersive (in purely elastic media). In other words, the speed of propagation is independent of frequency or wavenumber. Surface waves, by contrast, may be highly dispersive.
- Body waves transmit through media with continuously varying seismic properties $[\lambda(\mathbf{x}), \mu(\mathbf{x}), \rho(\mathbf{x})]$, but in the presence of seismic discontinuities, may yield reflected, transmitted or mode converted waves.

- The displacement produced by a harmonic wave in 1-D (e.g. a vibrating string) has the general form

  $$u(x,t) = A\exp[i(\omega t \pm kx)] = A\cos[i(\omega t \pm kx)] + Ai\sin[i(\omega t \pm kx)]$$

  Harmonic waves are characterised by displacement patterns that re-occur at regular intervals.
- The frequency, ν of a wave is defined as the number of oscillations (or cycles) per second, which is measured in Hertz (Hz).
- The period, T of a wave is simply the time required to complete one cycle: $T = 1/\nu$

Body waves

A few basic definitions....

- The angular frequency, ω, of a wave is defined as $\omega = 2\pi\nu = 2\pi/T$ (units of radians/s).
- The wavelength, η, of a wave is defined as the distance over which the wave pattern repeats itself.
- The angular wavenumber, k, of a wave is defined by $k = 2\pi/\eta$. The angular wavenumber is inversely proportional to the wavelength.
- The wavenumber, κ, of a wave is defined by $\kappa = 1/\eta = k/2\pi$, and is simply the inverse of the wavelength.
- Note that the wavespeed can therefore be expressed as:

$$v = \frac{\omega}{k} = \frac{\eta}{T} = \eta\nu$$

Plane harmonic waves

- To help understand the phenomenon of P and S waves, the concept of plane waves is introduced.
- For a plane wave, displacement only varies in the direction of wave propagation, and is constant in the directions orthogonal to the propagation direction.
- The P-wave scalar potential of a harmonic plane wave can be written:

  $$\Phi(\mathbf{x}, t) = iA\exp[i(\omega t \pm \mathbf{k} \cdot \mathbf{x})]$$

  where $\mathbf{x}$ is the position vector and $\mathbf{k}$ is the wavenumber vector.
- The direction of the wavenumber vector $\mathbf{k}$ gives the direction of wave propagation, and its magnitude is inversely proportional to wavelength: $|\mathbf{k}| = 2\pi/\eta$

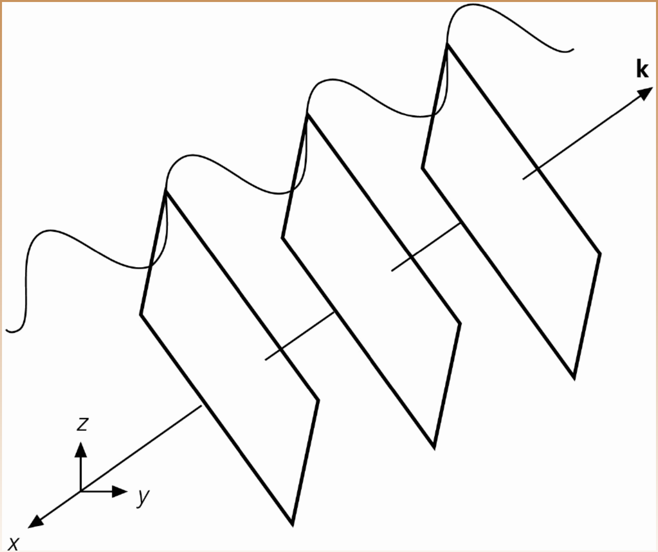
Plane harmonic waves

- Similarly, the S-wave vector potential of a harmonic plane wave can be written:

$$\mathbf{\Psi}(\mathbf{x}, t) = i\mathbf{B} \exp[i(\omega t \pm \mathbf{k} \cdot \mathbf{x})]$$

  noting that the amplitude coefficient **B** is now a vector.
- A plane wave is a wave in which the wavenumber vector **k** is independent of position and the surfaces of constant phase are planar and perpendicular to the direction of propagation.
- The propagation speed of a plane wave is defined by $|\mathbf{v}| = \omega/|\mathbf{k}|$.

Plane harmonic waves



- Using our definition of a P-wave scalar potential, the displacement field $\mathbf{u}_P$ is defined by:

$$\mathbf{u}_P = \nabla \Phi(\mathbf{x}, t) = \mp \mathbf{k} A \exp[i(\omega t \pm \mathbf{k} \cdot \mathbf{x})]$$

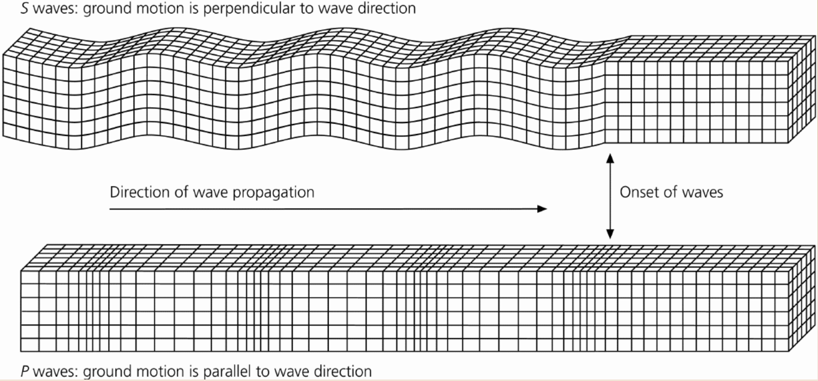
  Clearly, the displacement of a plane harmonic P-wave is parallel to **k**, the direction of propagation.

- Similarly, using our definition of an S-wave vector potential, the displacement field $\mathbf{u}_S$ is defined by:

$$\mathbf{u}_S = \nabla \times \mathbf{\Psi}(\mathbf{x}, t) = \mp \mathbf{k} \times \mathbf{B} \exp[i(\omega t \pm \mathbf{k} \cdot \mathbf{x})]$$

  In this case, the direction of displacement is perpendicular to the propagation direction **k** and the amplitude coefficient **B**. The projection of **B** onto the plane perpendicular to **k** defines a direction of polarisation for the S waves.

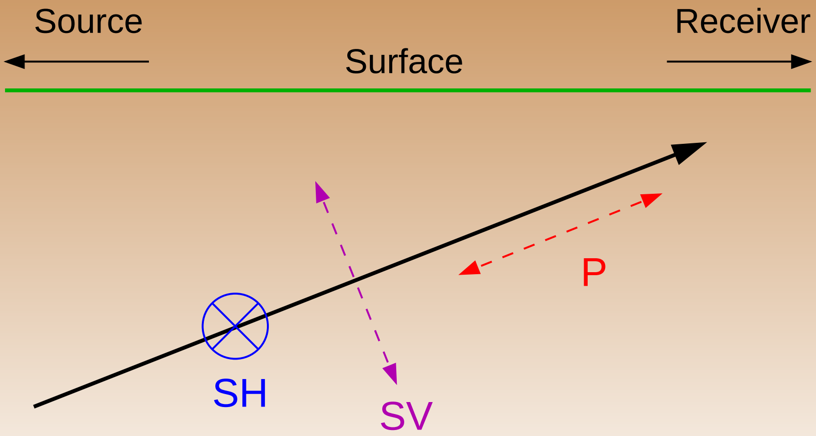
Plane harmonic waves
Displacement fields



SV and SH waves

- In real applications, the z axis often defines the vertical direction while the $x - z$ plane is oriented along the great circle connecting the source and receiver.
- SV: Shear waves with displacement in the vertical $x - z$ plane.
- SH: Shear waves with displacement in the horizontal $x - y$ plane.
- Although we could choose any two orthogonal polarisations in the plane of the shear wave displacements, using SV and SH is particularly convenient.

SV and SH waves



- Seismic waves transport both kinetic and elastic strain (or potential) energy as they propagate.
- Consider a harmonic plane SH wave travelling in the z direction with displacement in the y direction:

$$u_y(z,t) = B\cos(\omega t - kz)$$

- The kinetic energy in a volume V is the integral of the sum of the kinetic energy associated with each component of the displacement:

$$E_k = \frac{1}{2}\int_V \rho \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}V$$

noting that for a fixed mass m with velocity v, $E_k = \frac{1}{2}mv^2$.

- For the plane wave defined above, the volume integral reduces to a line integral. The kinetic energy per unit area of wavefront averaged over a wavelength η can therefore be written:

$$\begin{aligned}
E_k &= \frac{1}{2\eta}\int_0^{\eta} \rho \left(\frac{\partial[B\cos(\omega t - kz)]}{\partial t}\right)^2 \mathrm{d}V \\
&= \frac{1}{2\eta}\rho B^2\omega^2 \int_0^{\eta} \sin^2(\omega t - kz)\mathrm{d}z \\
&= \frac{1}{2\eta}\rho B^2\omega^2 \int_0^{\eta} \frac{[1-\cos 2(\omega t - kz)]}{2}\mathrm{d}z \\
&= \frac{1}{2\eta}\rho B^2\omega^2 \frac{\eta}{2} = \frac{B^2\omega^2\rho}{4}
\end{aligned}$$

- Consider a simple spring with a restoring force given by $f = kx$, where k is the spring constant. Compressing the spring by a distance dx requires work against the spring, equal to the integral of the applied force multiplied by the distance:

$$E_w = \int_0^x kx\mathrm{d}x = \frac{1}{2}kx^2$$

  This quantity equals the potential energy stored in the spring.

- By analogy, the strain energy stored in a volume is simply the integral of the product of the stress and strain components summed together:

$$E_w = \int \sigma_{ij} e_{ij} \mathrm{d}V = \int c_{ijkl} e_{ij} e_{kl} \mathrm{d}V$$

- The only non-zero strain components are:

$$e_{yz} = e_{zy} = \frac{1}{2}\frac{\partial u_y}{\partial z} = Bksin(\omega t - kz)/2$$

and the only non-zero stress components are:

$$\sigma_{yz} = \sigma_{zy} = 2\mu e_{zy} = \mu Bksin(\omega t - kz)$$

- The strain energy per unit area of wave front averaged over the wavelength in the propagation direction can therefore be written:

$$E_w = \frac{1}{2\eta}\int_0^{\eta} \mu B^2 k^2 \sin^2(\omega t - kz)dz = \frac{\mu B^2 k^2}{4} = \frac{B^2\omega^2\rho}{4}$$

Energy carried by a plane wave

- Note that $\mu k^2 = \rho w^2$ since from before the wavespeed $v = \sqrt{\mu/\rho} = \omega/k$
- The strain energy and kinetic energy averaged over a wavelength is therefore equal, and the total energy E_T is:

$$E_T = E_k + E_w = \frac{B^2 \omega^2 \rho}{2}$$

- The energy flux is the rate at which the wave transports energy past a fixed point. The average energy flux $\dot{E}$ is simply equal to the average energy multiplied by the wavespeed v:

$$\dot{E} = \frac{v B^2 \omega^2 \rho}{2}$$

Energy carried by a plane wave

- The total energy and flux are proportional to the square of the amplitude and the frequency. Thus, increasing the frequency with fixed amplitude will quadratically increase the energy carried by the wave, and vice versa.
- The energy flux provides insight into how waves behave when they change media. For example, as water waves travel into shallower water, their velocities decrease, so amplitude increases to conserve energy.
- Similarly, when seismic waves pass from bedrock into soft soil with lower velocity and density, their amplitudes increase.

P-SV reflection at a free surface

- In order to calculate what happens when a body wave interacts with the boundary between two media, we need to make use of two continuity conditions:
  - Displacement is continuous
  - Traction is continuous

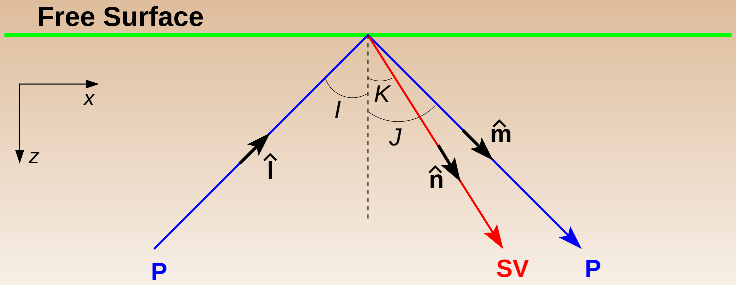
  Since both of these quantities are vectors, we imply continuity of all three components of each vector.
- A special case is a stress-free boundary or free surface, where a vacuum or material with low rigidity (e.g. the atmosphere) lies on one side of the boundary. In this case, the traction on the surface is effectively zero, so the three stress components are zero, but the displacement components are undefined.

P-SV reflection at a free surface

- Consider a P-wave propagating in the *xz* plane incident on a free surface $z = 0$.
- In this case, both P and S wave motion may be found in the reflected elastic energy, but no component of y displacement can appear because there is no y-displacement in the incident wave and no shear stress applied to $z = 0$.
- The resulting S-wave will be polarised with displacement only in the vertical plane perpendicular to **k**, the direction of propagation. In other words, it will be an SV-wave.

P-SV reflection at a free surface

- In order to quantify the behaviour of P-SV reflection at the free-surface, we define unit vectors in the direction of propagation of incident P ($\hat{\mathbf{l}}$), reflected P ($\hat{\mathbf{m}}$) and reflected S ($\hat{\mathbf{n}}$), which have angles of incidence I, J and K respectively.



P-SV reflection at a free surface

- The components of the direction vectors in terms of the incident angles are given by

$$\hat{\mathbf{l}} = (\sin I, 0, \cos I), \quad \hat{\mathbf{m}} = (\sin J, 0, -\cos J), \quad \hat{\mathbf{n}} = (\sin K, 0, -\cos K)$$

- Since the magnitude of an arbitrary wavenumber vector $\mathbf{k}$ is given by $|\mathbf{k}| = \omega/|\mathbf{v}|$, the three wavenumber vectors are $\mathbf{l} = \omega/\alpha\hat{\mathbf{l}}$, $\mathbf{m} = \omega/\alpha\hat{\mathbf{m}}$ and $\mathbf{n} = \omega/\alpha\hat{\mathbf{n}}$.
- Assume that the incident P wave has unit amplitude, and that the amplitude of the reflected P and S waves are B and C respectively.
- The displacement vectors for the two P-waves are parallel to the unit vectors, so the displacement functions for the incident and reflected P-waves are $\hat{\mathbf{l}} \exp[i(\mathbf{l} \cdot \mathbf{x} - \omega t)]$ and $B\hat{\mathbf{m}} \exp[i(\mathbf{m} \cdot \mathbf{x} - \omega t)]$ respectively.

P-SV reflection at a free surface

- The displacement associated with the reflected S wave is perpendicular to the direction of propagation $\hat{\mathbf{n}}$, and also must be perpendicular to $\hat{\mathbf{y}}$, a unit vector in the direction of the y-axis (since there is no initial displacement in the y-direction)
- The S-wave displacement function can therefore be written as $C(\hat{\mathbf{n}} \times \hat{\mathbf{y}}) \exp[i(\mathbf{n} \cdot \mathbf{x} - \omega t)]$.
- In order to calculate the relative amplitudes B and C of the two types of reflected wave, we need to sum the incident wave and two reflected waves, and then ensure that the free surface boundary conditions are satisfied.

P-SV reflection at a free surface

- The total displacement field is

$$\mathbf{u} = \hat{\mathbf{l}}\exp[i(\mathbf{l}\cdot\mathbf{x}-\omega t)] + B\hat{\mathbf{m}}\exp[i(\mathbf{m}\cdot\mathbf{x}-\omega t)] + C(\hat{\mathbf{n}}\times\hat{\mathbf{y}})\exp[i(\mathbf{n}\cdot\mathbf{x}-\omega t)]$$

- From this displacement field, we can calculate the strain components, and hence the stress or traction components.
- The four displacement gradients at the free surface ($z = 0$) are:

$$\frac{\partial u_x}{\partial z} = \frac{i\omega}{\alpha}\hat{l}_x\hat{l}_z\exp[i(l_x x-\omega t)] + B\frac{i\omega}{\alpha}\hat{m}_x\hat{m}_z\exp[i(m_x x-\omega t)] - C\frac{i\omega}{\beta}\hat{n}_z{}^2\exp[i(n_x x-\omega t)]$$

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P-SV reflection at a free surface

$$\frac{\partial u_z}{\partial x} = \frac{i\omega}{\alpha}\hat{l}_x\hat{l}_z \exp[i(l_x x - \omega t)] + B\frac{i\omega}{\alpha}\hat{m}_x\hat{m}_z \exp[i(m_x x - \omega t)] + C\frac{i\omega}{\beta}\hat{n}_x{}^2 \exp[i(n_x x - \omega t)]$$

$$\frac{\partial u_x}{\partial x} = \frac{i\omega}{\alpha}\hat{l}_x{}^2 \exp[i(l_x x - \omega t)] + B\frac{i\omega}{\alpha}\hat{m}_x{}^2 \exp[i(m_x x - \omega t)] - C\frac{i\omega}{\beta}\hat{n}_x\hat{n}_z \exp[i(n_x x - \omega t)]$$

$$\frac{\partial u_z}{\partial z} = \frac{i\omega}{\alpha}\hat{l}_z{}^2 \exp[i(l_x x - \omega t)] + B\frac{i\omega}{\alpha}\hat{m}_x{}^2 \exp[i(m_x x - \omega t)] + C\frac{i\omega}{\beta}\hat{n}_x\hat{n}_z \exp[i(n_x x - \omega t)]$$

P-SV reflection at a free surface

- Zero shear stress (zero horizontal traction) at $z = 0$ implies that

$$\sigma_{xz} = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = 0$$

- Substitution of the displacement gradient terms yields

$$2\hat{l}_x \hat{l}_z \exp[i(l_x x - \omega t)] + 2B\hat{m}_x \hat{m}_z \exp[i(m_x x - \omega t)] + \frac{C\alpha}{\beta}(\hat{n}_x{}^2 - \hat{n}_z{}^2)\exp[i(n_x x - \omega t)] = 0$$

which can only sum to zero if $l_x = m_x = n_x$.

- From before, if $l_x = m_x$, then $\omega \sin I/\alpha = \omega \sin J/\alpha$, so $I = J$. Thus, the angle of reflection for a P-wave equals the angle of incidence.

P-SV reflection at a free surface

- Since $l_x = n_x$, then $\omega \sin I/\alpha = \omega \sin K/\beta$, so

$$\sin I/\alpha = \sin K/\beta$$

  This means that the SV wave is reflected with a smaller angle of incidence than that of the reflected P-wave.
- The condition of zero shear stress has therefore produced one equation with two unknowns B and C:

$$2\hat{l}_x\hat{l}_z + 2B\hat{m}_x\hat{m}_z + \frac{C\alpha}{\beta}(\hat{n}_x{}^2 - \hat{n}_z{}^2) = 0$$

  which can also be written:

$$B\beta \sin 2I + C\alpha \cos 2K = \beta \sin 2I \qquad (1)$$

P-SV reflection at a free surface

- A second equation arises from the condition of zero normal stress at $z = 0$:

$$\sigma_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} = 0$$

- Substitution of the displacement gradient terms yields:

$$(\lambda + 2\mu)\left[\frac{\omega}{\alpha}\hat{l}_z{}^2 + B\frac{\omega}{\alpha}\hat{m}_z{}^2 + C\frac{\omega}{\beta}\hat{n}_x\hat{n}_z\right]$$
$$+\lambda\left[\frac{\omega}{\alpha}\hat{l}_x{}^2 + B\frac{\omega}{\alpha}\hat{m}_x{}^2 - C\frac{\omega}{\beta}\hat{n}_x\hat{n}_z\right] = 0$$

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P-SV reflection at a free surface

- Rearrangement of the above equation yields

$$B\left[(\lambda+2\mu)\hat{m}_z{}^2+\lambda\hat{m}_x{}^2\right]+C\frac{\alpha}{\beta}\left[(\lambda+2\mu)\hat{n}_x\hat{n}_z-\lambda\hat{n}_x\hat{n}_z\right]$$
$$=-\left[\lambda+2\mu)\hat{l}_z{}^2+\lambda\hat{l}_x{}^2\right]$$

- This can be simplified by using the fact that $\hat{m}_x{}^2=\sin^2 J$, $\hat{m}_z{}^2=\cos^2 J$, $\hat{l}_x{}^2=\sin^2 I$ and $\hat{l}_z{}^2=\cos^2 I$:

$$B\left[(\lambda+2\mu)-2\mu\hat{m}_x{}^2\right]+C\frac{\alpha}{\beta}\left[2\mu\hat{n}_x\hat{n}_z\right]=-\left[(\lambda+2\mu)-2\mu\hat{l}_x{}^2\right]$$

- In terms of angles, and using the relationships $\rho\alpha^2=(\lambda+2\mu)$ and $\rho\beta^2=\mu$:

$$B\left[\alpha^2-2\beta^2\sin^2 I\right]-C\alpha\beta\sin 2K=-\left[\alpha^2-2\beta^2\sin^2 I\right]\qquad(2)$$

P-SV reflection at a free surface

- Equations (1) and (2) are simultaneous equations for the two unknowns B and C, which can be solved to derive the reflection coefficients as a function of incidence angle I.
- For normal incidence $I = 0$, and so from Equation (1), $C = 0$. In other words, there is no conversion from P to S.
- In general, there is a solution with $B = 0$, i.e. total conversion from P to S, which occurs at an incident angle of around 50°.
- Incident SV is similarly converted into P and SV upon reflection. Conversion of energy from P to SV or vice versa is sometimes referred to as mode conversion.

P-SV reflection at a free surface

- When a wave is incident on a boundary that is not a free surface, the continuity conditions are more difficult to apply, because each incident wave gives rise to two upgoing and two downgoing rays.
- SH is always separate, however, because it has no common component of displacement with P or SV.
- Note that this treatment is strictly valid for isotropic media only; for general anisotropic media, up to three reflected and three transmitted waves can be generated for a single incident path.