PEAT8002 - SEISMOLOGY
Lecture 6: Ray theory

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- Here, we consider the problem of how body waves (P and S) propagate through a medium in which the elastic parameters vary with spatial location.
- The elastic wave equation in a medium with spatially variable properties is

$$\begin{aligned}\rho\ddot{\mathbf{u}} =& \nabla\lambda(\nabla\cdot\mathbf{u}) + \nabla\mu\cdot[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}] + (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) \\ &- \mu\nabla\times\nabla\times\mathbf{u}\end{aligned}$$

- The two terms containing $\nabla\lambda$ and $\nabla\mu$ mean that P and S motions do not decouple in heterogeneous media.

- However, if the scale length of variations in λ and μ are large compared to the seismic wavelength, then P and S can be treated separately and the elastic wave equation is simplified.
- Even so, solving the elastic wave equation requires exhaustive computational effort.
- Ray theory is an alternative approach in which a point on the wavefront is tracked rather than the complete wavefield.
- Ray theory is extensively used due to its simplicity, speed and applicability to a wide range of problems.

- Ray theory is strictly valid for media whose length scale variation of λ and μ is much larger than the seismic wavelength (the high frequency assumption).
- At low frequencies, diffraction and scattering can be significant, and ray theory is not generally valid.
- Ray theory is an integral part of many seismological techniques, including body wave tomography, migration of reflection data, and earthquake relocation.
- The process of tracking the kinematic evolution of seismic energy also brings with it the possibility of computing other wave-related quantities such as traveltime, amplitude, attenuation, or even the high frequency waveform, which can then be compared to observations.

- Under the high frequency assumption, the full wave equation can be greatly simplified. Here, we consider the propagation of P-waves in heterogeneous media.
- From before, the wave equation is:

$$\nabla^2 \Phi - \frac{1}{\alpha^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

  where Φ represents the scalar potential of a P-wave.
- Now assume a harmonic solution of the form:

$$\Phi = A(\mathbf{x}) \exp[-i\omega(T(\mathbf{x}) + t)]$$

  where $T(\mathbf{x})$ is a phase function which describes the arbitrary distribution in space of a surface of constant phase.

- If we take the gradient of the scalar potential then

$$\nabla\Phi = \nabla A\exp[-i\omega(T+t)] - i\omega A\nabla T\exp[-i\omega(T+t)]$$

- The divergence of the gradient of the scalar potential is thus

$$\begin{aligned}\nabla^2\Phi &= \nabla^2 A\exp[-i\omega(T+t)] - i\omega\nabla T\cdot\nabla A\exp[-i\omega(T+t)]\\ &\quad -i\omega\nabla A\cdot\nabla T\exp[-i\omega(T+t)]\\ &\quad -i\omega A\nabla^2 T\exp[-i\omega(T+t)]\\ &\quad -\omega^2 A\nabla T\cdot\nabla T\exp[-i\omega(T+t)]\end{aligned}$$

- The second derivative of Φ with respect to time is

$$\frac{\partial^2\Phi}{\partial t^2} = -\omega^2 A\exp[-i\omega(T+t)]$$

- Substitution of the above terms into the wave equation yields:

$$\nabla^2 A - i\omega\nabla T \cdot \nabla A - i\omega\nabla A \cdot \nabla T - i\omega A\nabla^2 T - \omega^2 A\nabla T \cdot \nabla T + \frac{\omega^2 A}{\alpha^2} = 0$$

- This can be rewritten as:

$$\nabla^2 A - \omega^2 A|\nabla T|^2 - i[2\omega\nabla A \cdot \nabla T + \omega A\nabla^2 T] = \frac{-A\omega^2}{\alpha^2}$$

- The above equation can be separated into its real and imaginary parts.

- **Real part:**

$$\nabla^2 A - \omega^2 A |\nabla T|^2 = \frac{-A\omega^2}{\alpha^2}$$

- Dividing through by $A\omega^2$ and taking the high frequency approximation yields the **eikonal equation**:

$$\boxed{|\nabla T|^2 = U^2}$$

where $U = \textit{slowness} = 1/\textit{velocity}$.

- The eikonal equation describes the kinematic propagation of high frequency waves.

- **Imaginary part:**

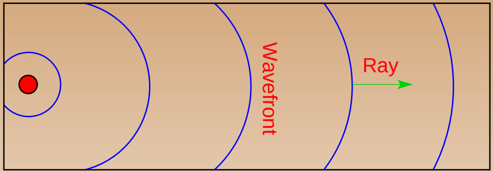
$$2\omega\nabla A \cdot \nabla T + \omega A\nabla^2 T = 0$$

- Dividing through by ω yields the **transport equation**:

$$\boxed{2\nabla A \cdot \nabla T + A\nabla^2 T = 0}$$

- The transport equation can be used to compute the amplitude of propagating waves.
- Substitution of the appropriate general S-wave vector potential into the elastic wave equation for an S-wave leads to identical expressions for the eikonal and transport equations. Thus, they are valid for any high frequency body wave.

- $T(\mathbf{x}) = constant$ defines surfaces called wavefronts.
- $\nabla T(\mathbf{x})$ defines raypaths.

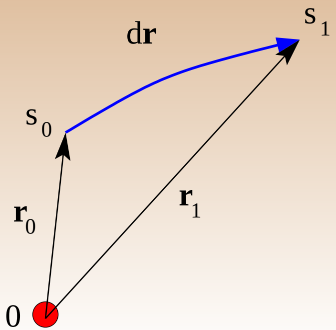


- The function $T(\mathbf{x})$ has units of time and simply represents the time required by the wavefront to reach $\mathbf{x}$ from some reference location $\mathbf{x_0}$.
- In fully anisotropic media, the eikonal and transport equations have a slightly more complex form due to the presence of the elastic tensor $\mathbf{c}$.

- If we denote s as the arc length parameter along a ray and $\mathbf{r}$ as the position vector of the ray, then

$$\frac{d\mathbf{r}}{ds} = \frac{\nabla T}{U}$$

since both $d\mathbf{r}/ds$ and $\nabla T/U$ are unit vectors parallel to the path ($d\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$ and $ds = |d\mathbf{r}|$).



- The rate of change of traveltime along the path is simply defined by the slowness, so

$$\frac{\mathrm{d}T}{\mathrm{d}s} = U$$

- If we take the gradient of both sides, then

$$\frac{\mathrm{d}\nabla T}{\mathrm{d}s} = \nabla U \qquad (1)$$

  noting the commutation of $\mathrm{d}/\mathrm{d}s$ and ∇.

- From before,

$$\nabla T = U\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \qquad (2)$$

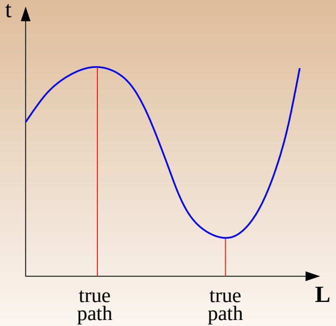
- Combining Equations (1) and (2) produces

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[U\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right] = \nabla U$$

  which is the kinematic ray equation and describes the trajectory of ray paths in smoothly varying isotropic media.
- It will be shown later how this equation can be reduced to forms suitable for initial and boundary value ray tracing.
- The ray equation requires U to be differentiable, and therefore is not applicable at the boundary between two media of different wavespeed.

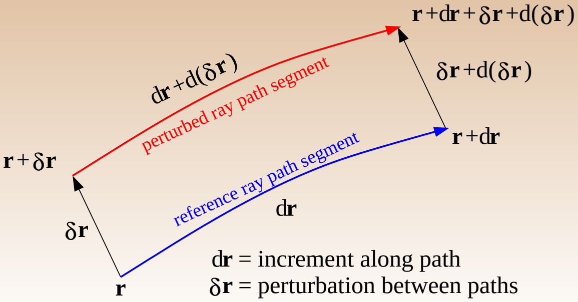
- Fermat's principle states that the ray path between two points P and Q is a path of stationary time

$$t_{PQ} = \int_P^Q U ds = \text{extremum}$$



- To prove Fermat's principle, we need to show that when a ray path is perturbed, the effect on traveltime is second order. A perturbation in the path perturbs the traveltime as follows:

$$\delta t_{PQ} = \delta \int_P^Q U ds = \int_P^Q \delta U ds + U \delta(ds)$$



- The first term in the integrand on the RHS is the contribution caused by a change in velocity; the second term is the contribution caused by the change in path length.
- If we first consider the change in path length,

$$
\begin{aligned}
\delta(ds) &= |\mathbf{dr} + d(\delta\mathbf{r})| - |\mathbf{dr}| \\
&= \sqrt{\mathbf{dr}\cdot\mathbf{dr} + 2\mathbf{dr}\cdot d(\delta\mathbf{r}) + d(\delta\mathbf{r}).d(\delta\mathbf{r})} - \sqrt{\mathbf{dr}\cdot\mathbf{dr}} \\
&= \sqrt{\mathbf{dr}\cdot\mathbf{dr} + 2\mathbf{dr}\cdot d(\delta\mathbf{r})} - \sqrt{\mathbf{dr}\cdot\mathbf{dr}} \\
& \quad \text{since} \quad |d(\delta\mathbf{r})| << |\mathbf{dr}| \\
&= ds\sqrt{1 + \frac{2\mathbf{dr}\cdot d(\delta\mathbf{r})}{ds^2}} - ds \\
& \quad \text{since} \quad ds = \sqrt{\mathbf{dr}\cdot\mathbf{dr}} \\
&= \frac{\mathbf{dr}}{ds}\cdot d(\delta\mathbf{r})
\end{aligned}
$$

- The last equality arises from the fact that:

$$\left[1 + \frac{d\mathbf{r} \cdot d(\delta\mathbf{r})}{ds^2}\right]^2 = 1 + \frac{2d\mathbf{r} \cdot d(\delta\mathbf{r})}{ds^2} + \frac{d(\delta\mathbf{r})^2}{ds^2}$$

where the last term on the RHS is ≈ 0.

- The other term in the integrand, δU, is simply:

$$\delta U = \nabla U \cdot \delta\mathbf{r} = \frac{\partial U}{\partial x}\delta x + \frac{\partial U}{\partial y}\delta y + \frac{\partial U}{\partial z}\delta z$$

where $\nabla U \cdot \delta\mathbf{r}$ is a directional derivative in $\delta\mathbf{r}$ direction.

- Combining these two results yields:

$$\delta t_{PQ} = \int_P^Q \left[\nabla U \cdot \delta \mathbf{r} ds + U \frac{d\mathbf{r}}{ds} \cdot d(\delta \mathbf{r}) \right]$$

- If we now apply integration by parts to the RHS term of the integrand:

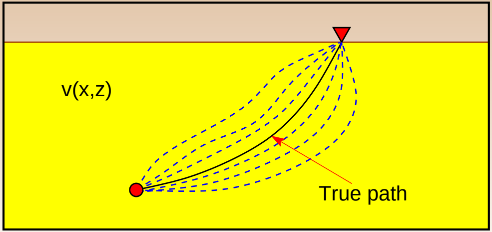
$$\int_P^Q U \frac{d\mathbf{r}}{ds} \cdot d(\delta \mathbf{r}) = \left[\delta \mathbf{r} \cdot U \frac{d\mathbf{r}}{ds} \right]_P^Q - \int_P^Q \delta \mathbf{r} \cdot \frac{d}{ds} \left(U \frac{d\mathbf{r}}{ds} \right) ds$$

  The first term on the RHS is zero since $\delta \mathbf{r} = 0$ at P and Q, the source and receiver.

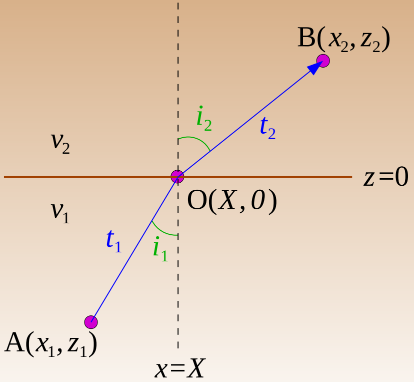
- Therefore, the perturbation becomes:

$$\delta t_{PQ} = \int_P^Q \left[\nabla U - \frac{d}{ds} \left(U \frac{d\mathbf{r}}{ds} \right) \right] \cdot \delta \mathbf{r} ds \quad (1)$$

- The integrand of the previous equation is zero by the wave equation.
- Therefore, we have shown that by ignoring higher order terms that the first-order perturbation of traveltime due to a perturbation in the ray path is zero, and we have proven Fermat's principle.



- We can use Fermat's principle to derive Snell's law, which describes the refraction of a ray path at an interface between media of different wavespeeds.



- The total traveltime T between A and B is given by

$$\begin{aligned} T &= t_1 + t_2 \\ &= \frac{AO}{v_1} + \frac{OB}{v_2} \\ &= \frac{\sqrt{z_1^2 + (X - x_1)^2}}{v_1} + \frac{\sqrt{z_2^2 + (X - x_2)^2}}{v_2} \end{aligned}$$

- From Fermat's principle, $\mathrm{d}T/\mathrm{d}X = 0$, so

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}X} &= \frac{1}{2}\frac{2(X - x_1)}{v_1\sqrt{z_1^2 + (X - x_1)^2}} + \frac{1}{2}\frac{2(X - x_2)}{v_2\sqrt{z_2^2 + (X - x_2)^2}} \\ &= \frac{X - x_1}{v_1\sqrt{z_1^2 + (X - x_1)^2}} + \frac{X - x_2}{v_2\sqrt{z_2^2 + (X - x_2)^2}} = 0 \end{aligned}$$

- From the plot, we have that:

$$\sin i_1 = \frac{X - x_1}{\sqrt{z_1^2 + (X - x_1)^2}}$$

$$\sin i_2 = \frac{x_2 - X}{\sqrt{z_2^2 + (X - x_2)^2}}$$

- We can now write Snell's law in its usual form:

$$\boxed{\frac{\sin i_1}{v_1} = \frac{\sin i_2}{v_2}}$$

- By combining Snell's law and the kinematic ray equation, it is possible to trace rays in the presence of 3-D laterally varying media that contain internal boundaries.