Electromagnetic damping of elastic waves: a simple theory

F. E. M. LILLEY[1] AND C. M. CARMICHAEL
Department of Geophysics, University of Western Ontario, London, Ontario

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An order-of-magnitude estimate is made to check the effect observed in previously described experiments in which vibrations in a metal bar were damped by an applied magnetic field that was strongly non-uniform. Though the awkward geometry of the experiment has prevented an accurate analytical solution of the problem, some reasonable assumptions allow a simple expression for the effect to be obtained directly. This expression is in agreement with the experimental results regarding the dependence of the effect upon frequency, bar dimensions, density, electrical conductivity, and magnetic field gradient.

Introduction

This note refers to a set of experiments, reported previously by the authors, in which the natural vibrations excited in a metal bar were attenuated by the application of a strongly non-uniform, but steady, magnetic field (Lilley and Carmichael 1968). The mechanism of attenuation is the common one of electrical eddy-current damping. As the part of the bar in the region of the magnetic field gradient vibrates, it experiences a changing applied magnetic field which induces a secondary field in the opposite sense. The secondary field is supported by electrical eddy currents in the bar. These interact with the applied magnetic field to give a Lorentz force opposing, and thus damping, the motion. The energy lost by the bar is dissipated as heat due to the resistive component of the eddy currents.

Although well-established laws of classical physics govern the effect, the geometry of the apparatus and of the applied magnetic field used in the experiments in question has proved too awkward for an analytical solution of the problem to be obtained. Some simplifications of the problem, however, which are reasonable upon physical grounds, enable a practical order-of-magnitude estimate of the effect to be made directly.

The problem is essentially concerned with a boundary layer effect. An allied problem, dealing with the same phenomenon taking place remote from boundaries, has been treated by Lilley (1967) and Lilley and Smylie (1968). The interest to geophysics of this physical process is its possible relevance to interaction between magnetic fields in the core of the earth and the free modes of elastic vibration of the earth.

The Problem Simplified

The complete problem requires finding solutions of the electromagnetic induction equation, and the equation of motion of the bar, for an applied magnetic field as generated between the poles of a "horse-shoe" electromagnet.

The first simplification derives from the fact that the magnetic non-uniformity is short compared to the wave-length of the vibrations of the bar. Thus the dominant energy loss occurs in those sections of the bar situated in the regions of magnetic field gradient; and, in these, the effect of strain can be neglected. This is equivalent to ignoring $(B.\nabla)v$ relative to $(v.\nabla)B$ in the expansion of curl $(v \times B)$ in the induction equation (Roberts 1967, p. 35, m.k.s. units),

$$[1] \qquad \frac{\partial B}{\partial t} = \mathrm{curl}\,(v \times B) + \frac{1}{\mu\sigma}\nabla^2 B$$

where v represents velocity, B magnetic field, μ permeability, σ electrical conductivity, and t is time. These points are demonstrated by the first two experiments described in Lilley and Carmichael (1968).

To estimate the energy loss, therefore, a model is taken of a rigid section of bar oscillating in a region of non-uniform field. The non-uniformity is idealized as one of uniform gradient, as shown in Fig. 1. For a bar that

[1]Present address: Department of Geophysics and Geochemistry, Australian National University, Canberra, Australia.

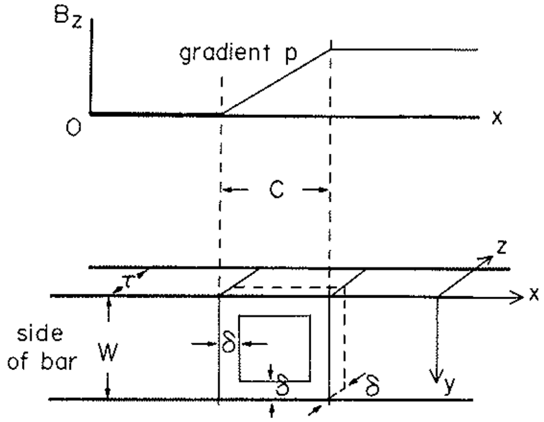

FIG. 1. Diagram showing the idealized applied magnetic field profile, (top), and the eddy current channel regions, (bottom).

maintains its harmonic motion during the attenuation of its vibrations, the attenuation time constant is given by (Knopoff 1964),

$$\gamma_m = \omega \Delta E / 4\pi E \tag{2}$$

where ΔE is the energy lost by the bar per cycle of vibration, E the total mechanical energy of the bar, ω the frequency of vibration, and γ_m the attenuation coefficient.

The bar is of length L, width W, thickness τ, and density ζ. The uniform magnetic gradient is applied across it for a distance C, as shown in Fig. 1. The section of the bar to which the gradient is applied oscillates with velocity v,

$$v = v_0 \sin \omega t$$

Consider the particular rectangular section of bar, of dimensions W by C, to which the gradient is applied. Within the rectangle, every point experiences a simultaneous change of applied magnetic induction given by $v_0 p \sin \omega t$, where p is the measure of the magnetic gradient. This is an important step. Because the currents flow in the bar, a transformation must be made to a coordinate system moving with the bar, and relative to this the applied field is not seen to move. What is observed at any point is the variation of the strength of the applied field with time. The time rate of change of applied flux through the rectangle is thus

$$\frac{d\phi}{dt} = CWv_0 p \sin \omega t$$

which is the same effect as if the bar were stationary, and an alternating field, uniform in space, were applied. Outside the rectangle, there is no change in applied flux sensed at any point in the bar.

The changing applied flux will induce an opposing secondary flux in the bar, supported by eddy currents, and the dissipation of energy by these eddy currents represents the energy loss by the bar as the vibrations are damped. The crucial problem is to find the eddy current distribution, and the accurate solution of this for the geometry in question has proved to be too difficult. However, from the variety of more simple problems in electromagnetic theory that have been solved rigorously, such as the incidence of an electromagnetic wave upon a plane conductor, and the flow of an alternating current in a wire, it is reasonable to predict that the eddy currents will flow around the perimeter of the rectangular section of bar, largely restricted to two paths, one on each side. The paths extend one electromagnetic "skin-depth" in from the boundary edges of the section, as shown in Fig. 1. The assumption is implied here that the skin-depth is small compared to the bar dimensions i.e.

$$\delta < W, \quad \delta < C, \quad \delta < \tau \tag{3}$$

where the skin-depth δ is given by

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

In like manner, an order-of-magnitude estimate of the energy dissipation of the eddy currents may be made, by calculating the energy which would be lost were the electromotive force causing the eddy currents to be applied steadily to the loops in which the eddy currents are channelled. By Faraday's law, the electromotive force causing the eddy currents around the perimeter will be of the same order as the change in flux through the area enclosed. Each loop, as drawn in Fig. 1, has resistance R,

$$R \sim \frac{2(C + W)}{\sigma\delta^2}$$

The power loss is then

$$\left(\frac{d\phi}{d}\right)^2 \Big/ R \sim \frac{\sigma\delta^2}{(C + W)} C^2 W^2 p^2 v_0^2 \sin^2 \omega t$$

TABLE I

Comparison of experiment and theory for the attenuation of standing waves by an applied field gradient, as characterized by the dependence of the attenuation constant, γ_m, upon the parameters listed

Parameter	Dependence observed	Theoretical dependence
Local velocity	Power of 2	Power of 2
Frequency	Power of −1.0 (brass) −1.1 (Al) −1.2 (Cu)	Power of −1
Length of bar	Power of −1	Power of −1
Thickness (3 different bars)	1:0.84:0.59	1:0.71:0.52 (inverse dependence)
Density (Al:Cu:brass)	1:0.23:0.31	1:0.30:0.31
Electrical conductivity	No dependence; brass shows anomalous behavior when conductivity condition fails. (See text.)	No dependence, under the conductivity conditions of the theory.
Field gradient	Power of 2	Power of 2

and the energy dissipation over one cycle is

$$\Delta E \sim \frac{2\pi(v_0 pCW)^2}{(C + W)\mu\omega^2}$$

The total mechanical energy in the bar is

$$E = \tfrac{1}{4}\zeta W L\tau V_0^2$$

for a bar hanging freely and vibrating in a longitudinal mode with maximum velocity V_0. Then, using [2],

$$\text{[4]} \qquad \gamma m \sim \frac{p^2}{\mu\omega\rho L\tau}\left(\frac{v_0}{V_0}\right)^2 \zeta$$

where

$$\zeta = 2C^2 W/(C + W)$$

a function of apparatus geometry only. If, in particular, the magnetic gradient is applied across an antinode of the bar,

$$v_0 = V_0$$

and

$$\gamma_m \sim \frac{p^2\zeta}{\mu\omega\rho\tau L}$$

An interesting feature of this equation is its independence of electrical conductivity, once the conditions [3], which involve conductivity, are satisfied.

Comparison with Experimental Results

The results of the experimental investigation of the effect are compared in Table I with the expression for the effect as obtained in the previous section of this note. That the agreement is quite good to some extent justifies the order-of-magnitude approach taken in deriving the expression [4]. The most interesting and diagnostic result is related to electrical conductivity, as restricted by the inequalities [3]. Of the lengths W, C, and τ the last is the least for the experiments described in Lilley and Carmichael (1968). The most restrictive condition is therefore

$$\delta < \tau$$

Experiment 3 of Lilley and Carmichael (1968) was carried out using aluminum, brass, and copper bars 3.2×10^{-3}m ($\frac{1}{8}$ in.) in thickness, in the frequency range 700 to 10 000 Hz. Some appropriate skin-depths are given in Table II. The table shows that the inequality is satisfied except for brass at frequencies below 1750 Hz. Inspection of Fig. 4 of Lilley and Carmichael (1968) shows quite clearly the different frequency dependence of the decay constant for brass in this range, where the inequality, involving electrical conductivity, is no longer satisfied.

Finally it should be noted that the uneven power (of minus one) by which the attenuation constant depends upon frequency, indicates that the damping is caused by a non-linear mechanism (Knopoff and MacDonald 1958). The damping mechanism in this case is the Lorentz force, F, opposing the motion. It is given, at any point, by

$$F = J \times B$$

where J is the local current density. Now J is related to B by

TABLE II

Some values of the electromagnetic skin-depth in the frequency range of the experiments

Frequency (Hz)	Skin-depth (m)		
	copper	aluminum	brass
700	2.5×10^{-3}	3.2×10^{-3}	5.1×10^{-3}
1750	1.6×10^{-3}	2.0×10^{-3}	3.2×10^{-3}
5000	9.5×10^{-4}	1.2×10^{-3}	1.9×10^{-3}

$$J = 1/\mu \text{ curl } B$$

and B is related to v by the induction eq. [1]. The dependence of F upon v may therefore be expected to be more complicated than that of simple linearity.

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