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THE ILL-POSED NATURE OF GEOPHYSICAL PROBLEMS

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§1. INTRODUCTION

Because of the natural interest of *homo sapiens* in his surroundings, much of what is now classified as geophysics formed the earliest science. For example, the geometry theorems of the ancient Greeks contributed to fundamental geophysics in the sense that they enabled the mensuration of areas of the earth's surface. Within the last several hundred years, right up until modern times, the progress of science generally has been closely associated with progress in earth and planetary physics: Gilbert's sixteenth century work on geomagnetism, and Newton's seventeenth century work on planetary dynamics, are two examples which come to mind. There has thus accumulated a considerable body of knowledge about the earth, much of it obtained in most significant experiments, such as the famous Cavendish experiment (1798) of "weighing the earth". In this exercise the gravitational constant G was determined, and thus knowing the radius of the earth and the acceleration of a falling body near its surface it was possible to estimate the mass of the earth, (a more recent determination of which is 5.98×10^{27} gram).

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A large part of modern solid-earth geophysics is now concerned with the variation and distribution of various physical parameters within the body of the earth. For example, a very relevant problem concerns the internal distribution of the earth's 5.98 x 10^{27} gram of total mass, mentioned above. Here direct observation is not possible, and the method of attack is to try to exploit some process which is in some way affected by the interior of the earth, to see if this process can give the information required. Some particular physical data are observed; and the problem is the inversion of these observations. A common first-order approximation in geophysics, and one which has fortunately shown itself to be reasonable in many instances, is that the interior of the earth is essentially spherically symmetric. Were this not actually so, the more complicated situation would have meant that much less progress would have been made.

This paper will largely be concerned with traditional studies of geophysics, the physics of which are in principle well understood. To keep a correct perspective however it should be mentioned that some of the most important modern geophysical theories depend on physical processes not fully understood: an example would be the long-term flow of rocks which on a shorter time scale behave as if rigid. An equally fascinating case is the use of classical physics (e.g. seismology) to probe the centre of the earth and in so doing to explore the physical behaviour of material there, which is at temperatures and pressures not at present within the range of laboratory apparatus.

§2. THE PARTIAL DIFFERENTIAL EQUATIONS OF BASIC GEOPHYSICS

The question thus arises as to what physical phenomena can the geophysicist observe on the surface of the earth, these phenomena to be dependent on some parameter or parameters inside the earth. In the analysis of such phenomena, which will shortly be listed, it is necessary of course to understand the basic physics involved. Fortunately most basic geophysical methods depend only on well-understood "classical" physics of the nineteenth century, and in fact nearly all are covered by the partial differential equation

$$A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial u}{\partial t} + C\nabla^2 u = D$$

where t denotes time, and the other symbols may vary with position in space; D may vary also with time. Depending on the values taken by A, B and C it is well known that this equation has three distinctive forms: called "elliptic", "parabolic" and "hyperbolic", in analogy with the three forms of the second order algebraic equation which describes conic sections. It is thus possible to classify the fields of basic geophysics according to which form of the differential they obey:

I. Elliptic; arising in the description of the gravity and magnetostatic potential fields.

(i) Laplace's equation, (homogeneous) $\nabla^2 u = 0$

(ii) Poisson's equation, (inhomogeneous) $\nabla^2 u = 4\pi G\rho$

u denotes either the gravity or magnetic potential; ρ is the density distribution of mass in space for the gravity case. In the magnetostatic case $G\rho$ is replaced by $\mathrm{div}\underset{\sim}{M}$, where $\underset{\sim}{M}$ is the magnetic dipole moment per unit volume.

II. Parabolic; arising in diffusion phenomena.

(i) Electromagnetic induction in the earth, (homogeneous)

$$\nabla^2 u - \mu\sigma \frac{\partial u}{\partial t} = 0$$

u represents a component of magnetic or electric field; μ denotes permeability and σ electrical conductivity.

(ii) The flow of heat in the earth, (inhomogeneous)

$$\nabla^2 u - \frac{1}{k}\frac{\partial u}{\partial t} = -\frac{A}{K}$$

u denotes local temperature, k thermal diffusivity, K thermal conductivity, and A the heat generated locally per unit time per unit volume.

III. Hyperbolic; the wave equation of seismology, (homogeneous).

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

where u denotes dilatation (for example), and v then represents the speed of propagation of a dilatational wave. It is important to note that although what is actually measured or sensed will usually be u, what is actually of interest are the other parameters involved. Hence the u-observations in a way form boundary conditions, which will give values (or restrict the ranges of the values) of the other parameters. Geophysical field work, and observatory practice, is essentially the business of collecting boundary conditions to be used when attempting to solve or model the appropriate differential equation as it applies within the earth.

The three types of partial differential equation given above may be soluble, that is, u expressed analytically as a function of space, time and the other parameters, if appropriate boundary conditions are known: for the homogeneous potential equation, the value of u around a closed edge; for the homogeneous diffusion equation, value or slope at one boundary, to enable projection forward in time; and for the homogeneous wave equation, value and slope at an open boundary in time. It is also important, of course, that the boundary conditions be given in some reasonable geometry. In basic geophysics the geometries taken are usually those of spherical symmetry, (if considering the earth on a global scale), or horizontal layering, (if considering the earth on a local scale). Major difficulties arise when departures from these simple symmetries are allowed. Given these two geometries, however, it is clearly possible for field geophysics to supply the boundary conditions needed at some points (but not all) of the boundary of the earth. The consequences of the incompleteness of the measurements of field geophysics will be discussed in §4. It is appropriate to first discuss a more profound difficulty: the lack of knowledge in some instances of the source terms involved.

§3. THE FIRST LEVEL OF NON-UNIQUENESS: UNKNOWN SOURCE FIELDS

The most fundamental difficulty in studying the geophysical phenomena given in the previous section arises when the appropriate equations are inhomogeneous, and the non-zero D term giving rise to the inhomogeneity is not known. For example, while the gravitational potential above the earth may be measured and expressed in terms of Laplace's equation, when it comes to geophysics u is not an end in itself: what is of interest is the term controlling the potential field within the earth according to Poisson's equation

$$\nabla^2 u = 4\pi G\rho \ .$$

In fact, it is a result of potential theory (for example, Kellogg [1]) that the interpretation of observed u to give a ρ-distribution is fundamentally non-unique: a factor which affects every aspect of gravity and magnetic interpretation. The non-uniqueness can be reduced if the density distribution is restricted to some particular model, such as a buried sphere, for example. But such models have to be justified on other grounds, and may beg the geophysical question. There are, however, certain reasonable ranges of models, the interpretation of data within which gives maxima or minima that are geologically useful (for example, Grant and West [2]). In practical geophysics the horizontal resolution of gravity surveys can also be very useful, their lack of vertical resolution notwithstanding.

The heat flow equation also contains an inhomogeneous source term; that allowing for the generation of heat within the earth. Only if this distribution and those of thermal conductivity and diffusivity are assumed can the profile of temperature with depth (the "geotherm") be estimated from measurements of the surface heat flow. The electromagnetic induction equation, from the point of view of the solid-earth geophysicist, avoids this dilemma; for its source term is external to the earth, arising from electric currents flowing in the ionosphere, (Matsushita and Campbell [3]). It is possible for the "primary" fields to be separated out, and the ambiguity in what may have caused them ignored.

The wave equation also avoids the source trouble, in that a seismological source, be it earthquake or explosion, can usually be regarded as instantaneous, and ended before the wave has travelled a significant distance. Thus seismic signals travel "source-free". The consequent avoidance of complication, and the possibilities of unique interpretation which result, have always appealed (and rightly so) to many geophysicists frustrated by the non-uniqueness of Poisson's equation. In the ranks of seismologists one finds many defaulted "potential-field" men.

§4. THE SECOND LEVEL OF NON-UNIQUENESS: INCOMPLETE AND IMPRECISE DATA

The previous two sections, in discussing under what circumstances a particular partial differential equation may have been soluble, were not considering any limitations in the data other than that they were restricted to observations on the surface of the earth. In practice, however, observed data will be both incomplete and imprecise; and in consequence interpretations made using them will be non-unique: even for phenomena which escape the unknown-source trouble of the previous section. No geophysical method escapes the difficulty of imperfect data.

It becomes of paramount importance then to know how wide is the uncertainty of a result obtained by the interpretation of certain data. This perhaps obvious statement deserves emphasis because in geophysics there has been a tendency to overlook it. Given measurements of u for any equation in §2, it is anything but straightforward to invert the data to give, (say) distributions of ρ, σ, or v. As has been mentioned, only in certain cases can an analytic expression be given for u in terms of the other parameters involved.

A certain tradition has therefore become established of "model fitting". A particular model of the earth will be chosen, perhaps arbitrarily, and its perhaps arbitrary parameters adjusted until the response of the model to the physical phenomenon agrees tolerably well with the response observed for the real earth. Thus the adjusted model is a "possible" earth; but nothing is known of all the other possible earths, or even how wide a range they may cover.

During the last decade a movement has started in an attempt to remedy this situation. It appears to have been commenced by Backus and Gilbert [4], and applies to geophysical phenomena which are Fréchet differentiable, and for which the Fréchet derivatives are known. In essence this restricts the method to those phenomena already mentioned for which u may be expressed analytically in terms of the other parameters, for the Fréchet derivatives are essentially derivatives of u (or whatever is the data parameter) in terms of the other parameters. The Backus-Gilbert formalism does not therefore apply to the many model-fitting procedures in geophysics which are more complicated, and which proceed by numerical methods simply because analytic solutions for u in terms of the various parameters of the model have never been obtained; (an example would be the electromagnetic response of an irregular-shaped three-dimensional body).

This is not, however, to underscore the Backus-Gilbert formalism, which is undoubtedly of great significance. It will now be outlined with reference to a simple hypothetical example.

<u>Note</u>. Other methods (than the approach of Backus and Gilbert) for the inversion of geophysical data, which take the problem of non-uniqueness directly into account, have been proposed. They include the use of Monte Carlo and linear programming procedures. The former have been examined by Press [5] and Anderssen, Cleary and Worthington [6], and the latter by Johnson [7].

§5. A SIMPLE EXAMPLE

Consider a simple earth model, specified by four parameters, as shown in Figure 1. An inner core of radius r transmits seismic signals with speed v_1; the outer part of the sphere, of radius R, transmits seismic signals with speed

v_2. Assume for the sake of this example that r and R are known, so that there are two unknown parameters of geophysical interest: v_1 and v_2. Consider one observational datum to be known: the travel time T of a seismic signal between points an angular distance θ apart, (as shown).

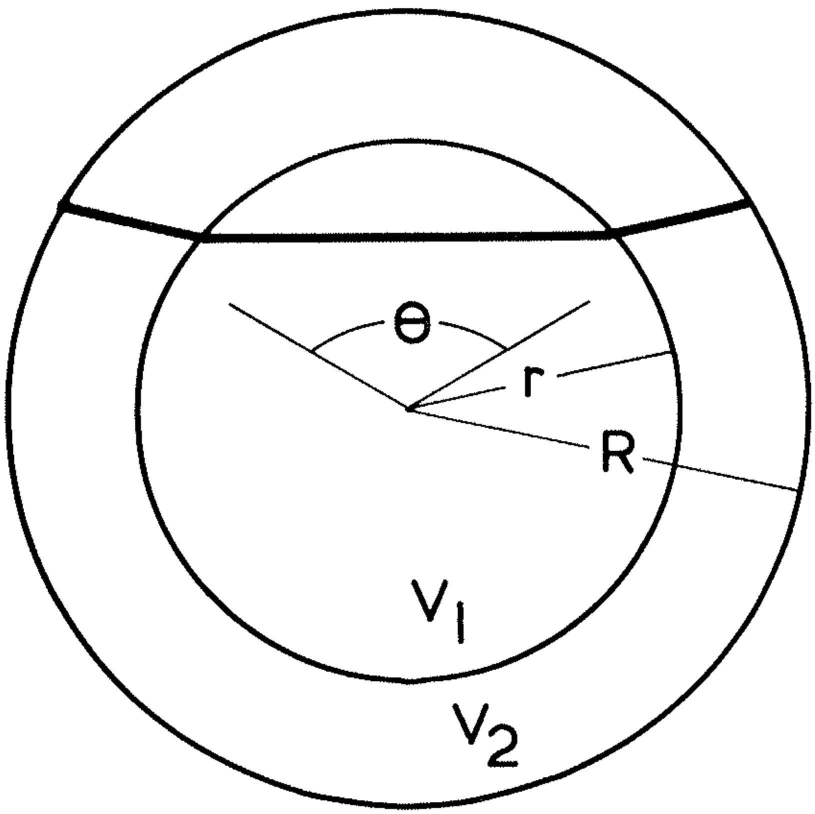

FIG. 1

In such a case, T may be expressed analytically in terms of r, R, v_1, v_2 and θ, (see, for example, Bullen [8]) and so it is possible to imagine a surface in T, v_1 and v_2 space (for given r, R and θ) as in Figure 2.

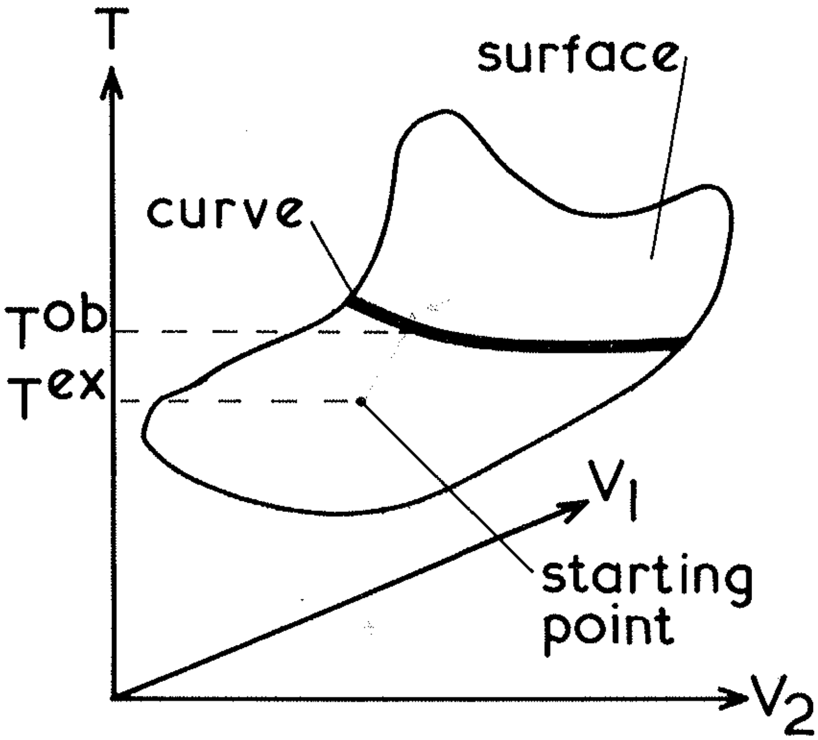

FIG. 2

Thus any one observation of T will define a plane which will generally cut the surface in a curve. Any point along this curve is an acceptable "eartn", and so the existence of the curve represents the ill-posed non-unique nature of the problem. The Backus-Gilbert method does not remove this non-uniqueness; but if for some reason an approximate model of what is expected is known, then the method finds the point on the curve nearest the "expected" model.

It is not necessary to compute the whole surface. Rather, the T for the expected model is computed: say this is T^{ex}. Then the discrepancy between the expected T^{ex} and the observed T^{ob} may be expressed

$$T^{ob} - T^{ex} = \delta T = \left(\frac{\partial T}{\partial v_1}\hat{v}_1 + \frac{\partial T}{\partial v_2}\hat{v}_2\right).(\delta v_1 \hat{v}_1 + \delta v_2 \hat{v}_2) \qquad (5.1)$$

where $\delta v_1 = v_1^{ob} - v_1^{ex}$, $\delta v_2 = v_2^{ob} - v_2^{ex}$, that is, the δv_i represent the adjustments necessary to the "expected" v_i^{ex} to produce the acceptable v_i^{ob}. (The relation as just expressed implies that the surface is essentially planar between the "expected" point and the curve; if this is not so a number of iterations following this procedure may be necessary.)

The term $\left(\frac{\partial T}{\partial v_1}\hat{v}_1 + \frac{\partial T}{\partial v_2}\hat{v}_2\right)$ is the Fréchet derivative of the observable datum T with respect to the parameters of the model, and the model has been chosen so that this is known. Equation (5.1) as it stands can be satisfied by a wide range of $(\delta v_1, \delta v_2)$ values, each pair representing a different position along the curve of acceptable models. The criterion that the final model should be closest to the "expected" model is now invoked, and by a variational technique the particular $(\delta v_1, \delta v_2)$ pair satisfying equation (5.1) is found subject to $(\delta v_1^2 + \delta v_2^2)$ being a minimum. Starting from a given model, it is thus possible to move onto the curve from a starting point off the curve, given only that there are no local maxima or minima in between. The procedure has not removed the non-uniqueness represented by the curve; but it has produced the acceptable model closest to the starting model, and the necessity of computing the whole surface (or representative parts of it) has been avoided.

The next reward of the procedure is the rather more subtle one of resolution. In the example just given, the Fréchet derivative expresses the all-important information on how sensitive is the parameter being observed to changes in the physical parameters inside the earth. Clearly if T is a little sensitive to (say) v_1, then v_1 may be expected to be poorly resolved: a small change in δv_2 would allow a great change in δv_1, and in Figure 2 this would be apparent by the surface being rather flat in the v_1 direction. If, on the other hand, T was highly sensitive to v_1, but not sensitive to v_2 at all, then it would be reasonable to expect v_1 to be resolved near-perfectly, and v_2 not resolved at all.

To take this point of resolution further, consider now many travel-time data T_i, at different distances θ_i. Consider the earth model to consist of many (radially symmetric) layers, so that the speed at radius r is $v(r)$. Then, by extension of the procedure already followed for the simple case given above, after choosing a starting model and computing its response times the discrepancy in the i-th time may be written

$$\delta T_i = \int_0^1 K_i(r)\delta v(r)\, dr \tag{5.2}$$

where $K_i(r)$ is the Fréchet derivative, a function of radius just as the speed now is. (The radius of the surface, R, has been normalized to unity.) The extension of the previous example is now to satisfy equation (5.2), subject to $\int_0^1 [\delta v(r)]^2\, dr$ being a minimum. But consider, next, arbitrary multiplying functions $a_i(r_0)$, where r_0 is some particular point along the radius r, employed to give:

$$\sum_i a_i(r_0)\delta T_i = \int_0^1 \left[\sum_i a_i(r_0)K_i(r)\right]\delta v(r)\, dr\ .$$

If it so happened that

$$\sum_i a_i(r_0)K_i(r) = \delta(r - r_0) \tag{5.3}$$

where $\delta(x)$ is the Kronecker delta function, then

$$\sum_i a_i(r_0)\delta T_i = \delta v(r_0) \ ;$$

which would give the adjustment $\delta v(r_0)$ to be made to the speed distribution at the r_0 radial point to bring the model into exact agreement with the data, (as far as the r_0 radial point was concerned), and would mean, in effect, that the observed data were such that a combination of them had been found which was sensitive only to the speed at radius r_0 . In other words, the resolution at the radial point r_0 would be near-perfect, and there would be no necessity to resort to any criterion, like minimizing any integral, to remove the ambiguity at that point.

Usually, of course, equation (5.3) will not hold even given the freedom of choice of the $a_i(r_0)$ functions, and the best that can be done is to (say) minimize

$$\int_0^1 [\delta(r - r_0) - A(r,r_0)]^2 \, dr$$

where

$$A(r,r_0) = \sum_i a_i(r_0)K_i(r) \ .$$

The $A(r,r_0)$ function is thus a type of "resolution function", indicating how sensitive the phenomenon is to changes in the parameter at and around the r_0 radial position. The closer $A(r,r_0)$ can be made to approach the delta function $\delta(r - r_0)$, the more precise is the possible resolution of the speed parameter at r_0 .

This section has attempted to give an introduction to the Backus-Gilbert formalism in only the simplest terms. Altogether the relevant papers on the subject form a formidable series: Backus and Gilbert ([4], [9] and [10]). There are valuable introductions to the method also in the papers of Parker [11], Jackson [12] and Johnson and Gilbert [13]. Wiggins [14] examines the structure of the general linear inverse problem and a method for its solution, and shows how they relate to the work of Backus and Gilbert. For an introduction to the mathematical language in which the whole matter is couched, an excellent reference is Lanczos [15].

§6. CONCLUSIONS

Non-uniqueness in basic geophysics arises at two levels. The first level is a consequence of unknown source terms in the partial differential equations governing certain geophysical phenomena, and cannot be avoided, no matter how accurate and complete are the observational data. The second level affects all geophysical inversion interpretations, no matter what the partial differential equations involved, and is a consequence of the fact that observed geophysical data are never either perfectly accurate or complete.

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