Elastic Wave Motion and a Nonuniform Magnetic Field in Electrical Conductors[1]

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An applied magnetic field will perturb an elastic wave that is passing through an electrical conductor. Nonuniformity in the applied field introduces an extra dissipative effect that is not present for a uniform applied field. In this paper one case of the interaction of elastic wave motion with a nonuniform field is solved by means of series expansions in the two dimensionless perturbation parameters involved. The effect of the nonuniform field dominates the effect of the uniform field in regions where the local field strength is less than half of the change in field strength per wavelength. For seismic waves in the core of the earth, the effect would be most important for long-wavelength standing modes of free vibration. However, taking accepted values for the conductivity and field strength in the core, the effect of the nonuniform field is found to be insignificant, as is the uniform field effect.

INTRODUCTION

The elementary principles of electromagnetic theory predict that an applied magnetic field will perturb an elastic wave that is passing through an electrical conductor, because the motion associated with the wave will give rise to induced fields and currents, and there will be some energy dissipated. The quantitative estimate of this perturbation followed from the generalization of Alfvén's treatment of magnetohydrodynamic waves to include mediums that were compressible and were of finite conductivity [*Herlofson*, 1950; *van de Hulst*, 1951; *Anderson*, 1953; *Baños*, 1955].

The interaction of longitudinal wave motions with an applied magnetic field has interested geophysicists because this interaction must take place when seismic waves pass through the core of the earth where the electrical conductivity is high and the magnetic field, as predicted by dynamo theory, is strong. The question is whether the interaction will affect the seismic waves, or whether it is negligible. The problem is treated by *Knopoff* [1955], *Baños* [1956], *Knopoff and MacDonald* [1958], and *Kraut* [1965]. These authors conclude that the effect is indeed insignificant, but it is important to note that their estimates of attenuation are based on theory for a uniform applied field.

The magnetic field in the earth's core cannot, of course, be uniform everywhere, and it is the purpose of this paper to develop the theory for a particular case of the interaction of elastic wave motion with a nonuniform field to find the order of magnitude of the extra attenuation involved. A nonuniform field may be expected to introduce extra energy loss because, as the wave passes, the translation of the material will also induce currents in addition to the compression-expansion cycle that causes the energy loss in a uniform field.

The problem is initially one in magnetoelasticity. A general nonuniform magnetic field is difficult to treat mathematically, and the few previous publications on wave motion in a nonuniform field have all assumed the more simple case of infinite conductivity [*Gajewski and Winterberg*, 1963, 1965; *Uberoi*, 1964; and *Plato*, 1964]. However, we must consider the conductivity to be finite for energy loss to occur at all. We treat the most simple case of a nonuniform field in rectangular Cartesian coordinates, and, having found the result for plane waves propagating in an infinite medium, we shall apply it to seismic waves in the earth's core. This paper does not take into account the possible extra energy loss close to the boundaries

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of the medium, which has been shown in experiments by *Lilley and Carmichael* [1968] to be appreciable under some conditions.

Statement of Problem

We wish to investigate a special case of magnetoelastic wave motion in a nonuniform applied field $\mathbf{B}_0$ which is of the form $\mathbf{B}_0 = \hat{z}\, B_0\,(x)$. We seek wave motion solutions for an infinite, plane, longitudinal wave that is traveling in the x direction, perpendicular to the applied field. This particular case has been chosen for the convenience of having all derivatives zero with respect to y and z. Because the motion associated with the wave is in the direction of increasing field strength and cuts across the field lines, the effect of the nonuniformity should be clearly evident.

Equilibrium conditions. The magnetic field that has been specified imposes the condition that an equilibrium current density $\mathbf{J}_0$ must be present,

$$\mathbf{J}_0 = \frac{1}{\mu}(\nabla \times \mathbf{B}_0) = -\hat{y}\,\frac{1}{\mu}\frac{dB_0}{dx}$$

where μ represents permeability and the displacement current is taken as zero. There will be a Lorentz force interaction $\mathbf{L}_0$ between this equilibrium current density and the magnetic field given by

$$\mathbf{L}_0 = \mathbf{J}_0 \times \mathbf{B}_0 = -\hat{x}\,\frac{B_0}{\mu}\frac{dB_0}{dx}$$

Because the medium is elastic, strain will occur until this equilibrium Lorentz force is opposed exactly by the elastic restoring force at each point. Let us denote the resulting equilibrium displacement by $\mathbf{U}_0\,(x)$.

The current density must have associated with it an electric field $\mathbf{E}_0$ such that for no motion $\mathbf{J}_0 = \sigma\,\mathbf{E}_0$, where σ is the electrical conductivity. From Maxwell's second equation we have the equilibrium condition that $\nabla \times \mathbf{E}_0 = 0$; hence,

$$-\hat{z}(1/\sigma\mu)(d^2B_0/dx^2) = 0$$

The condition that the problem be one dimensional with y and z derivatives zero therefore produces the constraint that $dB_0/dx = p$ where p is a constant, and the inhomogeneous field of our problem is restricted to one of uniform gradient. We write it as $\mathbf{B}_0 = \hat{z}\,(px + B_2)$ where B_2 is the field strength at the origin, which, without loss of generality, we can take to be zero.

The equilibrium Lorentz force may now be expressed as

$$\mathbf{L}_0 = -\hat{x}(p^2x/\mu)$$

and the resulting strain in the medium, $\epsilon_x(x)$, is given by

$$\epsilon_x(x) = -\frac{1}{\rho_1\alpha_1^{\,2}}\int L_0\,dx = \frac{1}{\rho_1\alpha_1^{\,2}}\left(\frac{p^2x^2}{2\mu} + \text{constant}\right)$$

where ρ_1 and α_1 are the density and primary elastic velocity of the medium before the application of the field. Since we are treating an infinite medium, we can arbitrarily choose the strain at the origin to be zero, and the constant vanishes from the last expression. If the density of the medium is now described by $\rho(x)$, we have

$$\frac{\rho(x)}{\rho_1} = 1 - \frac{p^2x^2}{2\mu\rho_1\alpha_1^{\,2}} \qquad (1)$$

to order $(\epsilon_x)^1$ for small strains.

The general form of this equilibrium condition, from the theory of elasticity, is

$$-\mathbf{F}_0 = \alpha^2\nabla\nabla\cdot\mathbf{U}_0 - \beta^2\nabla\times\nabla\times\mathbf{U}_0$$

where $\mathbf{F}_0$ is the body force per unit mass and α and β are the primary and secondary elastic velocities. For Lorentz body forces only, $\mathbf{L}_0 = \rho\,\mathbf{F}_0$.

Disturbance of equilibrium. The equations have the general form

$$\mathbf{J} = \frac{1}{\mu}(\nabla \times \mathbf{B})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{L} = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$-\mathbf{F} = \alpha^2\nabla\nabla\cdot\mathbf{U} - \beta^2\nabla\times\nabla\times\mathbf{U} - \frac{\partial^2\mathbf{U}}{\partial t^2}$$

where $\mathbf{v}$ denotes particle velocity.

We consider a small disturbance, such that departures from the equilibrium values are small, and adopt the following notation for the disturbed quantities:

Total magnetic induction	$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$
Electric field	$\mathbf{E} = \mathbf{E}_0 + \mathbf{e}$
Current density	$\mathbf{J} = \mathbf{J}_0 + \mathbf{j}$
Lorentz force per unit volume	$\mathbf{L} = \mathbf{L}_0 + \mathbf{l}$
Particle displacement	$\mathbf{U} = \mathbf{U}_0 + \mathbf{u}$
Body force per unit mass	$\mathbf{F} = \mathbf{F}_0 + \mathbf{f}$

The equilibrium equations subtract from the general set above to give the following set, which have been linearized by ignoring the products of small quantities:

$$\mathbf{l} = \frac{1}{\mu}\left[(\nabla\times\mathbf{B}_0)\times\mathbf{b} + (\nabla\times\mathbf{b})\times\mathbf{B}_0\right] \quad (2)$$

$$-\mathbf{f} = \alpha^2\nabla\nabla\cdot\mathbf{u} - \beta^2\nabla\times\nabla\times\mathbf{u} - \frac{\partial^2\mathbf{u}}{\partial t^2} \quad (3)$$

$$\mathbf{j} = \sigma(\mathbf{e} + \mathbf{v}\times\mathbf{B}_0) \quad (4)$$

Upon taking the curl of this last equation, we obtain

$$\frac{1}{\sigma\mu}\nabla\times\nabla\times\mathbf{b} = -\frac{\partial\mathbf{b}}{\partial t} + \nabla\times(\mathbf{v}\times\mathbf{B}_0)$$

of Cartesian resolutes

$$\nabla^2 b_x\sigma\mu\,\partial b_x/\partial t = 0$$

$$\nabla^2 b_y - \sigma\mu\frac{\partial b_y}{\partial t} = 0$$

$$\nabla^2 b_z - \sigma\mu\frac{\partial b_z}{\partial t} = \sigma\mu\frac{\partial}{\partial x}(v_x B_0)$$

The first two are simple diffusion equations, and thus b_x and b_y may be taken as zero, as they have no source terms to sustain them. Only the equation for b_z has a source term and is coupled with the particle motion. Consequently, the disturbance magnetic field is always parallel to the applied field, and, like it, is a function of x alone.

If we now introduce a time dependence of e^{-iwt} for all perturbation quantities, the last equation may be written as

$$\frac{d^2 b_z}{dx^2} = -i\sigma\mu w\left[b_z + \frac{d}{dx}(u_x B_0)\right] \quad (5)$$

The x resolutes of (2) and (3) are

$$l_x = -\frac{1}{\mu}\frac{d}{dx}(b_z B_0)$$

$$-f_x = \alpha^2\frac{d^2u_x}{dx^2} + w^2 u_x$$

and, because, for Lorentz body forces only, $l_x = \rho f_x$, we have

$$\frac{d^2u_x}{dx^2} + \frac{w^2}{\alpha^2}u_x - \frac{1}{\mu\rho\alpha^2}\frac{d}{dx}(b_z B_0) = 0 \quad (6)$$

The pair of coupled equations 5 and 6 must now be solved for u_x and b_z to give the wave propagation.

At this point, a note may be made regarding the linearization of (2) to (4) above, which required that $\mathbf{B}_0 > \mathbf{b}$. At the origin, where $\mathbf{B}_0$ is zero, this condition is not satisfied. However, the amplitude of $\mathbf{b}$ depends upon the amplitude of vibration, which may be taken as being arbitrarily small. Therefore the range of nonlinearity is reduced to an arbitrarily small region about the origin. Elsewhere the linearity approximation is valid.

Solution by Expansion in Perturbation Series

Taking the case where the applied field is given by $B_0 = px$, we wish to solve (5) and (6) for u_x and b_z. We reduce the equations to dimensionless form and seek solutions that are series expansions in terms of the two perturbation parameters that become evident.

Taking u_{max} as the greatest amplitude of particle displacement that will occur at the origin, we define dimensionless variables

$$u = \frac{u_x}{u_{max}} \qquad b = \frac{b_z}{p u_{max}}$$

and now replace x by a dimensionless variable ξ, where $\xi = (xw/\alpha_1)$. The density variation (1) may now be expressed

$$\rho(\xi)/\rho_1 = 1 - (\xi^2/2q) \quad (7)$$

to order $(1/q)^1$, where q is the number

$$q = \mu\rho_1 w^2/p^2$$

Equations 6 and 5 may be written

$$\frac{d^2u}{d\xi^2} + u = \frac{1}{q}\left[\frac{d}{d\xi}(b\xi) + \frac{\xi^2 u}{2}\right] \quad (8)$$

$$b + \frac{d}{d\xi}(u\xi) = \frac{i}{s}\frac{d^2b}{d\xi^2} \quad (9)$$

where s is the other perturbation number

$$s = (\alpha_1^2\sigma\mu/w).$$

The parameter s has a simple relationship with the ratio of the elastic wavelength λ to the electromagnetic skin depth δ, given by $s = (1/2\pi^2)(\lambda^2/\delta^2)$.

The parameter q gives a measure of the relative size of the elastic phase velocity and the increase in Alfvén velocity over a typical distance. Thus $q = (\alpha_1^2/V_k^2)$, where V_k is the magnetohydrodynamic wave velocity in a field strength of $p\lambda/2\pi$.

In the case we are considering, the elastic wave is only slightly perturbed and the conductivity is very high, so that both s and q are large. In particular, in what follows, we assume

$$1 > \frac{1}{s} > \frac{1}{q} > \frac{1}{s^2} > \frac{1}{sq} > \frac{1}{q^2} \cdots$$

and seek solutions of the form

$$u = u_0 + \frac{u_1}{s} + \frac{u_2}{q} + \frac{u_3}{s^2} + \frac{u_4}{sq} + \cdots \qquad (10)$$

$$b = b_0 + \frac{b_1}{s} + \frac{b_2}{q} + \frac{b_3}{s^2} + \frac{b_4}{sq} + \cdots \qquad (11)$$

These expressions are substituted into (8) and (9), and coefficients of similar powers in the perturbation quantities are equated. The following set of simple differential equations is obtained, the members of which are solved consecutively.

Equating coefficients of unperturbed terms:

$$(d^2u_0/d\xi^2) + u_0 = 0$$

$$b_0 = -\frac{d}{d\xi}(u_0\xi)$$

of solution, taking the positively traveling wave,

$$u_0 = e^{i\xi} \qquad (12)$$

$$b_0 = -(1 + i\xi)e^{i\xi} \qquad (13)$$

These solutions represent the limiting case of infinite conductivity with the magnetic field so weak that the elastic wave is not perturbed at all. The following terms represent the perturbation of this limiting case.

Equating coefficients of $1/s$:

$$(d^2u_1/d\xi^2) + u_1 = 0$$

$$b_1 = i\frac{d^2b_0}{d\xi^2}$$

of solution $u_1 = 0$, $b_1 = (3i - \xi)e^{i\xi}$.

Equating coefficients of $1/q$:

$$\frac{d^2u_2}{d\xi^2} + u_2 = \frac{d}{d\xi}(\xi b_0) + \frac{\xi^2}{2}u_0$$

$$b_2 = -\frac{d}{d\xi}(u_2\xi)$$

of solution

$$u_2 = \left(\frac{i}{8}\xi - \frac{3}{8}\xi^2 - \frac{i}{4}\xi^3\right)e^{i\xi}$$

$$b_2 = \left(-\frac{i\xi}{4} + \frac{5\xi^2}{4} + \frac{11i\xi^3}{8} - \frac{\xi^4}{4}\right)e^{i\xi}$$

Equating coefficients of $1/s^2$:

$$(d^2u_3/d\xi^2) + u_3 = 0$$

$$b_3 = i(d^2b_1/d\xi^2)$$

of solution $u_3 = 0$, $b_3 = (5 + i\xi)e^{i\xi}$.

Equating coefficients of $1/sq$:

$$(d^2u_4/d\xi^2) + u_4 = \frac{d}{d\xi}(b_1\xi)$$

$$b_4 = -\frac{d}{d\xi}(u_4\xi) + i\frac{d^2b_2}{d\xi^2}$$

of solution $u_4 = [(\xi/2) + i\xi^2 - (\xi^3/6)]e^{i\xi}$, and so on. As would be expected, u_1 and u_3 are zero, because the propagation of the elastic wave is not affected merely by the existence of finite electrical conductivity.

Collecting the different terms in (10) and (11), we have the solutions

$$u = \left[1 + \frac{1}{q}\left(\frac{i\xi}{8} - \frac{3}{8}\xi^2 - \frac{i}{4}\xi^3\right) + \frac{1}{sq}\left(\frac{\xi}{2} + i\xi^2 - \frac{\xi^3}{6}\right)\right]e^{i\xi} \qquad (14)$$

with terms in the bracket to order $1/sq$, and

$$b = \left[-1 - i\xi + \frac{3i}{s} - \frac{\xi}{s}\right]e^{i\xi} \qquad (15)$$

with terms in the bracket to order $1/s$, which includes all the terms we shall need in this paper. To the same respective orders of accuracy, the amplitudes of u and b are

$$|u| = 1 - \frac{3}{8}\frac{\xi^2}{q} + \frac{\xi}{2sq} - \frac{\xi^3}{6sq} \qquad (16)$$

$$|b| = \left(1 + \xi^2 - \frac{4\xi}{s}\right)^{1/2} \tag{17}$$

The physical interpretation of the different terms in (14) is facilitated by considering the unit wave number of u_0 in (12) to be perturbed by $(k_1 + ik_2)$, where both k_1 and k_2 are small and real. We would then have

$$u = e^{i\xi + ik_1\xi - k_2\xi}$$
$$= e^{i\xi}(1 + ik_1\xi - k_2\xi)$$

to the first order of k_1 and k_2 within the parentheses. On comparison, we see that the imaginary perturbation terms in (14) may be interpreted as a perturbation in wave number, given by

$$k_1 = \frac{1}{8q} - \frac{\xi^2}{4q} + \frac{\xi}{sq}$$

so that the phase velocity with magnetic field applied is, to the same order of accuracy,

$$\alpha = \alpha_1\left(1 - \frac{1}{8q} + \frac{\xi^2}{4q} - \frac{\xi}{sq}\right)$$

Because the definition of ξ involves the wave number, there is thus a slight dispersion effect, the group velocity C being given by

$$C = \alpha_1\left(1 - \frac{1}{8q} + \frac{3\xi^2}{4q} - \frac{2\xi}{sq}\right) \tag{18}$$

Similarly the real perturbation terms in (14) may be interpreted as an attenuation factor given by

$$-k_2 = -\frac{3\xi}{8q} + \frac{1}{2sq} - \frac{\xi^2}{6sq}$$

to order $1/sq$.

It is interesting to examine these terms one by one. The first, which is dominant, does not represent attenuation through energy dissipation because the finite conductivity, which controls dissipation, is not involved. Rather it represents energy transferred to the induced magnetic field. The next term $1/2sq$ is in fact an amplifying effect, and we shall see that, as far as energy flux is concerned, it is compensated by variation in the local group velocity of the wave. Both these terms have resulted from the presence of the gradient in the applied field. The third term, which, as it occurs in (16), may be expressed

$$-\frac{\xi^3}{6sq} = \frac{-B_0^2(x)w^2x}{6\alpha_1^5\sigma\mu^2\rho_1}$$

has its origin in the same mechanism that causes dissipation in a uniform field. It may be compared with Knopoff's attenuation factor for an applied field of strength B_2, which, to the same order of accuracy, is

$$u_0 \exp\left(\frac{-B_2^2w^2x}{2\alpha_1^5\sigma\mu^2\rho_1}\right) = u_0\left(1 - \frac{B_2^2w^2x}{2\alpha_1^5\sigma\mu^2\rho_1}\right)$$

At any particular value of x, the rate of attenuation, $d|u|/dx$ is the same for both factors.

Two waves, $-e^{i\xi}$ and $-i\xi e^{i\xi}$, are present in (13) for b_0, and the terms in b_1 may be interpreted as being perturbations in the phase of each, due to the presence of finite conductivity. That is, to order $1/s$ within the bracket, (15) is represented by

$$b = -e^{(i\xi - 3i/s)} - i\xi e^{(i\xi - i/s)}$$

Energy Flux

To estimate the dissipative effects of the applied field, it is necessary to consider the energy flux carried by the wave, that is, the energy density of the disturbance times the group velocity. Three types of energy are present, kinetic energy, potential energy stored in the elastic medium, and magnetic energy. The interflow of the wave energy between these different forms is complicated; however, if we consider separately the mean mechanical wave energy per unit volume $\frac{1}{2}\rho\,|\mathbf{v}|^2$ and the mean magnetic wave energy per unit volume $(1/4)\,|\mathbf{b}|^2$, we may make an estimate of the energy flow by adding the flux due to each.

Taking (7), (16), and (18) into account, the mechanical energy flux is

$$S_1(\xi) = \tfrac{1}{2}\rho_1\alpha_1w^2u_{\max}^2 \cdot\left(1 - \frac{1}{8q} - \frac{\xi^2}{2q} - \frac{\xi}{sq} - \frac{\xi^3}{3sq}\right) \tag{19}$$

to order $1/sq$ within the parentheses. Using (17), the magnetic energy flux may be expressed

$$S_2(\xi) = \tfrac{1}{2}\rho_1\alpha_1w^2u_{\max}^2\left(\frac{1}{2q} + \frac{\xi^2}{2q} - \frac{2\xi}{sq}\right) \tag{20}$$

The total flux is therefore

$$S(\xi) = \tfrac{1}{2}\rho_1\alpha_1 w^2 u_{max}^2\left(1 + \frac{3}{8q} - \frac{3\xi}{sq} - \frac{\xi^3}{3sq}\right) \quad (21)$$

The perturbation of the energy flux by the magnetic field is now evident within the parentheses on the right-hand side of the last equation. The term $3/8q$ indicates that at the origin, where the applied field strength is zero, the wave carries more energy than a pure elastic wave of the same amplitude, owing to the presence of the field gradient. The term $-(3\xi/sq)$ is the dissipative effect of the nonuniform field which we have been seeking, and it may be compared with the remaining term $-(\xi^3/3sq)$ noted above as being the term resuting from the 'uniform field' effect. For a typical value of ξ of unity, the gradient effect is an order of magnitude stronger than the uniform field effect. The gradient effect dominates over a region extending to $3/2\pi$ (roughly ½) of a wavelength on either side of the origin. Outside this region, the uniform field effect is dominant.

We also note that, since the region is nonuniform, some energy will in general be lost from the forward-traveling wave by reflection. Such loss will be taken into account by our method of solution and, if significant, will be evident as a decrease of the energy flux with distance. However, in the limit of infinite conductivity ($s \to \infty$), the total energy flux as expressed in (21) does not vary with ξ, thus in the present problem the energy reflected is less than the order of accuracy of (21). Because negligible energy is reflected in the limiting case of infinite conductivity, terms that are present as a result of finite conductivity represent energy dissipated, not energy reflected. The result that negligible energy is reflected is consistent with the condition on the problem that the elastic wave be only slightly perturbed.

Conclusion

The method of using series in the perturbation parameters s and q has provided direct solutions to an order of accuracy known immediately. The energy flux of the mechanical wave is seen in (19) to be reduced as the wave progresses by three factors: the transfer of energy to build up the perturbation magnetic field, a dissipative effect due to the presence of the field gradient, and a dissipative effect arising from the 'uniform field' mechanism. Then in (20) some of the energy transferred from the mechanical wave to the magnetic wave is lost because of the diffusion possible in a material of finite conductivity. The dissipative effects due to the nonuniformity of the field are thus evident.

Finally, can we say anything more about seismic waves in the core? Knopoff and Kraut have shown that the effect of a uniform field on seismic waves in the core is negligible. The particular nonuniform field we have considered has a stronger effect than a uniform field within half a wavelength of the origin. The importance of the nonuniform field effect will therefore be most pronounced at long wavelengths, which in the seismic case arise from the free oscillations of the earth. However, the extremely small effect found by Kraut is not changed significantly by an increase of 1 order of magnitude, and the gradient effect found here for an infinite medium is negligible in the core of the earth.

Acknowledgments. We thank Dr. J. H. Blackwell, professor of applied mathematics in the University of Western Ontario, for much patient discussion. This paper was written while one of us (F.E.M.L.) was visiting the Department of Geodesy and Geophysics at Cambridge. He thanks Professor Sir Edward Bullard for allowing him to use the facilities of the department, and he is indebted to the National Research Council of Canada, which provided support through an overseas postdoctorate fellowship.

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(Received April 13, 1968;
revised June 10, 1968.)