

LINEAR RELATIONSHIPS IN GEOMAGNETIC VARIATION STUDIES

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The linear properties of Maxwell's equations enable the primary and secondary fields of geomagnetic variations to be related together by an induction tensor. Variations of total field which are confined on or near a plane, as described first by Parkinson, can occur formally only under certain conditions on either the induction tensor (that it have a simple degeneracy), or on the primary variation field components (that they consistently obey the same linear relationship over all time). The second condition is implausible for typical transient variations observed on earth, and examination of the first condition demonstrates that for variations with no consistent correlation of the horizontal components, the existence of a well-determined Parkinson vector means that observed vertical variations are secondary fields, induced by observed horizontal variations. This conclusion has important implications for the theoretical explanation of the geomagnetic "coast effect".

That primary fields be spatially uniform is a tacit condition of much theoretical work on geomagnetic variations, though the condition can in fact be relaxed to require spatial uniformity of the primary fields just within some "induction region". If however the primary fields are not uniform over the induction region a poor Parkinson-vector determination may result. This may be most noticeable at sites which have their induction regions greatly extended by a process such as current channelling.

1. Introduction

In studies of geomagnetic variations, it is common to seek empirical relationships between different components which are observed to be obeyed independently of time. This paper explores the theoretical significance of such linear relationships. A crucial question which arises is the possible correlation between, and the relative importance of, the vertical and horizontal components of the primary field variations, which originate external to the earth.

The paper is in three parts. In the first part, the general theoretical approach is taken that all three primary field components are a priori independent and of equal importance. In the second part, the simplest observational evidence for the existence of more restrictive conditions on the primary and secondary components is investigated. In the third part, the spatial uniformity of the primary fields is

considered, and the concept of an induction region is introduced.

An equation of central importance to this paper will be introduced as:

$$\Delta Z_T = A \Delta X_T + B \Delta Y_T$$

where all quantities may be complex. The notation will be explained at the appropriate place below. This equation will be termed "Parkinson's relation", as it expresses a linear relationship first observed by Parkinson (1959). Strictly speaking, Parkinson confined his attention to the case where all quantities were real: the equation would then specify a plane in a three-dimensional $\Delta X_T, \Delta Y_T, \Delta Z_T$ real space.

2. The general case for a spatially uniform source field

Electromagnetic induction phenomena of the type

observed in geomagnetic studies are taken to be governed by Maxwell's equations. These equations are linear, and thus certain relationships between inducing (primary) and induced (secondary) fields may be expressed directly and simply.

Consider primary magnetic variations originating external to the earth which induce, due to the presence of electrically conducting material within the earth, secondary variations at some particular point. For changes in the primary field which are spatially uniform over an indefinitely large region, the linearity of the induction process demands that if a particular primary component is to induce a particular secondary component, then the relationship between the two must be linear. For variations at a particular frequency, denote the components of the primary field by ΔX_P , ΔY_P , ΔZ_P and of the secondary field by ΔX_S , ΔY_S , ΔZ_S . Then secondary variations may be expressed in terms of primary variations by:

$$\begin{pmatrix} \Delta X_S \\ \Delta Y_S \\ \Delta Z_S \end{pmatrix} = \begin{pmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{pmatrix} \begin{pmatrix} \Delta X_P \\ \Delta Y_P \\ \Delta Z_P \end{pmatrix}$$

where, for example, the part of the Y -component of the secondary field induced by a particular change of ΔX_P in the primary field is given by $\Delta Y_S = k_{yx} \Delta X_P$.

The magnetic field components are complex quantities, with in-phase and quadrature parts, and the elements k_{ij} of the induction matrix are complex constants (or possibly zero). The matrix is in fact a tensor, because each suffix refers to one of a set of axes, and the matrix elements have appropriate transformation properties under rotation of the axes (Jeffreys and Jeffreys, 1956). The induction tensor, as it will now be called, is of order two.

For solid-earth geophysicists, the determination of such an induction tensor is the basic objective in observing geomagnetic variations, as the elements of the tensor are governed by the electrical conductivity structure of the earth. The considerable problem of inverting the results of the observations to give a conductivity distribution – see, for example, Bailey (1970) – is not entered upon in this paper: suffice it to note that the most that can be achieved from observation is the determination of some of the elements of the matrix.

If several instruments (preferably a two-dimensional array) are recording concurrently over a region, it is often advantageous to consider the observed field as the sum of a regional uniform part, of components ΔX_N , ΔY_N and ΔZ_N , and a locally varying anomalous part, of components ΔX_A , ΔY_A and ΔZ_A . For this case Schmucker (1970) has defined a different matrix, which in his notation is:

$$\begin{pmatrix} \Delta X_A \\ \Delta Y_A \\ \Delta Z_A \end{pmatrix} = \begin{pmatrix} h_h & h_d & h_z \\ d_h & d_d & d_z \\ z_h & z_d & z_z \end{pmatrix} \begin{pmatrix} \Delta X_N \\ \Delta Y_N \\ \Delta Z_N \end{pmatrix} \quad (1)$$

and Schmucker shows how in principle each element of this induction matrix may be determined.

Though simultaneous observations with many instruments is clearly preferable, much work has been done and may well continue to be done using just one instrument at a time. The purpose of this paper is to examine what conditions on the induction tensor or the source fields can be deduced from single-station operation, and in particular to determine what is implied by a good fit of observed data to Parkinson's relation.

Denoting the total field variations by ΔX_T , ΔY_T , ΔZ_T , so that $\Delta X_T = \Delta X_P + \Delta X_S$ etc., it follows that:

$$\begin{pmatrix} \Delta X_T \\ \Delta Y_T \\ \Delta Z_T \end{pmatrix} = \begin{pmatrix} k'_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k'_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k'_{zz} \end{pmatrix} \begin{pmatrix} \Delta X_N \\ \Delta Y_N \\ \Delta Z_N \end{pmatrix} \quad (2)$$

where $k'_{xx} = k_{xx} + 1$, and similarly for k'_{yy} and k'_{zz} . For single station operation it is common to seek a best fit of data to the equation:

$$\Delta Z_T = A \Delta X_T + B \Delta Y_T \quad (3)$$

(termed Parkinson's relation, as explained in the introduction), which may be written:

$$[A, B, -1] \begin{pmatrix} \Delta X_T \\ \Delta Y_T \\ \Delta Z_T \end{pmatrix} = 0$$

This would imply, referring to eq. 2, that:

$$[A, B, -1] \begin{pmatrix} k'_{xx} & k'_{xy} & k'_{xz} \\ k'_{yx} & k'_{yy} & k'_{yz} \\ k'_{zx} & k'_{zy} & k'_{zz} \end{pmatrix} \begin{pmatrix} \Delta X_P \\ \Delta Y_P \\ \Delta Z_P \end{pmatrix} = 0 \quad (4)$$

If ΔX_P , ΔY_P and ΔZ_P are to be able to vary independently, the only solution of eq. 4 would be:

$$[A, B, -1] \begin{pmatrix} k'_{xx} & k'_{xy} & k'_{xz} \\ k'_{yx} & k'_{yy} & k'_{yz} \\ k'_{zx} & k'_{zy} & k'_{zz} \end{pmatrix} = [0, 0, 0]$$

That is:

$$\left. \begin{aligned} A &= \frac{k'_{yy} k'_{zx} - k'_{yx} k'_{zy}}{k'_{xx} k'_{yy} - k'_{xy} k'_{yx}} \\ B &= \frac{k'_{xx} k'_{zy} - k'_{xy} k'_{zx}}{k'_{xx} k'_{yy} - k'_{xy} k'_{yx}} \end{aligned} \right\} \quad (5)$$

with the extra condition specified that:

$$k'_{xx} k'_{yz} k'_{zy} + k'_{yy} k'_{zx} k'_{xz} + k'_{zz} k'_{xy} k'_{yx} - k'_{xy} k'_{yz} k'_{zx} - k'_{yx} k'_{xz} k'_{zy} - k'_{xx} k'_{yy} k'_{zz} = 0 \quad (6)$$

In the general treatment taken so far there is no reason to assume that the special condition, eq. 6, will be satisfied, and therefore in the most general terms eq. 3 is not a fundamental equation of magnetic variations.

3. Significance of the determination of a linear relationship

In the previous section it was shown that a linear relationship $\Delta Z_T = A \Delta X_T + B \Delta Y_T$ is not a result of general basic theory. Such relationships have, however, been determined as "best fits" at many different places on the earth's surface, which must mean that at these places a certain condition (or conditions) holds which was not considered in the general theory of the previous section. This section examines the two possible conditions which would cause such a linear relationship to be observed. The simplification of a spatially uniform field is kept; the consequence of departures from spatial uniformity will be discussed in the next section.

There are two special circumstances, either of which will allow eq. 4 to be satisfied. The first is that the matrix:

$$\begin{pmatrix} k'_{xx} & k'_{xy} & k'_{xz} \\ k'_{yx} & k'_{yy} & k'_{yz} \\ k'_{zx} & k'_{zy} & k'_{zz} \end{pmatrix} \quad (7)$$

should have a simple degeneracy (which will automatically satisfy eq. 6); and the second is that ΔX_P , ΔY_P and ΔZ_P should no longer be independent; rather, one of ΔX_P , ΔY_P and ΔZ_P becomes a linear function of the other two. A good fit of data to eq. 3 indicates that one of these conditions is being approached: an appeal must be made to the known physics of geomagnetic variations to determine which, if either, is more likely. Both possibilities will now be examined in greater detail.

3.1. Degeneracy of the matrix

The first possibility, that of a simple degeneracy, when the rows (columns) of the matrix are linearly connected by a single relation, would mean that the matrix (7) could be expressed in the form (see Frazer et al., 1963, p.20):

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \begin{bmatrix} g & h & j \\ k & l & m \end{bmatrix} \equiv \begin{pmatrix} ag + dk & ah + dl & aj + dm \\ bg + ek & bh + el & bj + em \\ cg + fk & ch + fl & cj + fm \end{pmatrix}$$

where, on the left hand side, the columns of the first matrix (rows of the second) are linearly independent. The quantities a, b, c, \dots, l, m are complex.

The solutions for A and B are then:

$$A = \frac{ce - bf}{ae - bd} \quad \text{and} \quad B = \frac{af - cd}{ae - bd} \quad (8)$$

which are equivalent to eq. 5. These solutions (eq. 8) are not in themselves very informative about the induction matrix, and if more information on the elements of the matrix is to be obtained from observed estimates of A and B , other conditions on the field components must be invoked.

One possible and important source of extra information is the degree of correlation observed between ΔX_T and ΔY_T . If these two variation fields show no persistent correlation, as is often the case over an ensemble average of transient events, then $|k_{xy}| \ll 1$

and $|k_{yx}| \ll 1$ and the approximation may reasonably be made that:

$$k_{xy} = k_{yx} = 0$$

$$\text{i.e.: } bg + ek = ah + dl = 0$$

$$\text{so that: } gb = -ek \text{ and } ah = -dl$$

Under these circumstances the solutions (eq. 8) reduce to:

$$A = \frac{k_{zx}}{k'_{xx}} \text{ and } B = \frac{k_{zy}}{k'_{yy}}$$

so that eq. 3 may be expressed:

$$\Delta Z_T = \frac{k_{zx}}{k'_{xx}} (\Delta X_S + \Delta X_P) + \frac{k_{zy}}{k'_{yy}} (\Delta Y_S + \Delta Y_P)$$

leading to:

$$\Delta Z_T = k_{zx} \Delta X_P + k_{zy} \Delta Y_P + \left(\frac{k_{zx} k_{xz}}{k'_{xx}} + \frac{k_{zy} k_{yz}}{k'_{yy}} \right) \Delta Z_P \quad (9)$$

Note that k'_{zz} is then specified as:

$$k'_{zz} = \frac{k_{zx} k_{xz}}{k'_{xx}} + \frac{k_{zy} k_{yz}}{k'_{yy}} \quad (10)$$

so that:

$$k_{zz} = -1 + \frac{k_{zx} k_{xz}}{k'_{xx}} + \frac{k_{zy} k_{yz}}{k'_{yy}} = -1 + Ak_{xz} + Bk_{yz}$$

However, for $k_{xy} = k_{yx} = 0$, ΔX_T and ΔY_T may be expressed:

$$\Delta X_T = k'_{xx} \Delta X_P + k_{xz} \Delta Z_P$$

$$\Delta Y_T = k'_{yy} \Delta Y_P + k_{yz} \Delta Z_P$$

and observed independence of ΔX_T and ΔY_T requires further that:

$$k'_{xx} \Delta X_P \gg k_{xz} \Delta Z_P \text{ and } k'_{yy} \Delta Y_P \gg k_{yz} \Delta Z_P \quad (11)$$

otherwise ΔX_T and ΔY_T would be correlated, in as much as both would correlate strongly with ΔZ_P . (For the general geometry being considered there would be no justification in appealing to just one of these inequalities.) Eq. 9 and 10 may thus be expressed:

$$\Delta Z_T \approx k_{zx} \Delta X_P + k_{zy} \Delta Y_P \text{ and } k'_{zz} \approx 0$$

This means that ΔZ_P induces a secondary vertical field which opposes the primary, and thus the observed vertical variation resulting from ΔZ_P is approximately zero. These circumstances are a clear statement that the ΔZ_T of eq. 3 is predominantly secondary, induced by the horizontal primary fields.

3.2. Possible interdependence of the components of the primary field

The second possible cause of Parkinson's relation, that one of ΔX_P , ΔY_P and ΔZ_P is linearly dependent on the other two over all time would give:

$$(Ak'_{xx} + Bk_{yx} - k_{zx}) \Delta X_P + (Ak_{xy} + Bk'_{yy} - k_{zy}) \Delta Y_P + (Ak_{xz} + Bk_{yz} - k'_{zz}) \Delta Z_P = 0 \quad (12)$$

In fact ΔX_P , ΔY_P and ΔZ_P must all be derivable from a scalar potential ΔW , so that they are interdependent to the extent that they must obey:

$$\Delta X_P = \frac{\partial \Delta W}{\partial x}, \quad \Delta Y_P = \frac{\partial \Delta W}{\partial y}, \quad \Delta Z_P = \frac{\partial \Delta W}{\partial z} \quad (13)$$

However eq. 13 does not require the satisfaction of eq. 12, and for different transient events such as magnetic storms, ΔW will vary as a function of time. A consistent explanation of Parkinson's relation on this basis does not seem possible, for if there is no consistent repeatability of the ionospheric currents causing such transients, then over the ensemble average of events ΔX_P and ΔY_P will be independent. This would be contrary to the fact that a well-established Parkinson vector, formed from an ensemble of events by a consistent correlation of ΔZ_P with some horizontal component of the primary field, would require a consistent $\Delta X_P/\Delta Y_P$ ratio to satisfy eq. 12.

Alternatively if the assumption is made that there is some repeatability in the primary fields of successive events, due to the general tendency of the ionospheric currents of magnetic substorms to extend from auroral regions into mid-latitudes, then the resulting fixed ratio of $\Delta X_P/\Delta Y_P$ at any latitude would require the Parkinson vector to point consistently in one direction over large areas, contrary to the well-observed property of Parkinson vectors to be normal to coast lines.

3.3. Conclusion to section 3

The only realistic and general interpretation of

Parkinson's relation is concluded to be a simple degeneracy of the induction matrix, not a time-independent correlation of the components of the primary horizontal field. The Parkinson relation therefore shows that primary horizontal variations are inducing vertical variations. Three of the nine matrix elements (7) will be zero: k_{yx} , k_{xy} and k'_{zz} . If in the inequalities (11) ΔX_P , ΔY_P and ΔZ_P are all of the same order, then k_{xz} and k_{yz} are also small, and the Parkinson values A and B give ratios between the remaining elements; $A = k_{zx}/k'_{xx}$, and $B = k_{zy}/k'_{yy}$.

In previous published work on the coast effect, the notion that horizontal fields induce vertical fields at a coastline has been common, but not unanimous. Edwards et al. (1971) for example, take a different approach: having obtained Parkinson vectors on the single instrument basis described above, they appear to assume a restrictive condition on ΔX_P and ΔY_P in explaining the coast effect observed. It is difficult to see how the induced fields of the theoretical section of their paper will correlate with the primary horizontal fields to give good Parkinson vectors: a rather special dependence of ΔZ_P on ΔX_P and ΔY_P would be required, and this dependence would have to vary from observing station to observing station in a somewhat improbable manner.

Similarly, the calculations of Bullard and Parker (1971), invoking variations in ΔZ_P as the cause of anomalous ΔZ_T at ocean edges, do not explain why in the case of bays and magnetic storms this ΔZ_T should correlate so well with ΔX_T and ΔY_T . The existence of Parkinson vectors, approximating the relationship examined, seems indeed to be irrefutable evidence that the coast effect for this type of variation is caused by horizontal field variations. For Parkinson vectors in essence do not correlate ΔZ_T merely with ΔX_S and ΔY_S , but with ΔX_T and ΔY_T , which include ΔX_P and ΔY_P . (This particular question has also been discussed recently by Hyndman and Cochrane (1971), with similar conclusions.)

4. Relaxation of the spatially uniform field condition and the concept of an induction region

In the case of practical geomagnetic variations, the primary field changes are not uniform over an indefi-

nately large region. In practice, however, consideration may be confined only to that space in which the induction taking place is important. At any instant, the secondary magnetic field B_S at an observing site is given by an integration following the Biot-Savart law (Stratton, 1941), notably:

$$B_S = \frac{\mu_0}{4\pi} \int_V \frac{\sigma(\mathbf{r}) \mathbf{E}(\mathbf{r}) \times \mathbf{r}}{|\mathbf{r}|^3} d\mathbf{v} \quad (14)$$

Here μ_0 is the permeability of free space, and $\sigma(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ are the electrical conductivity and electric field at the point \mathbf{r} relative to the observing site. In such an integration conduction only is relevant, and the volume V over which the integration is taken (which might be termed a "conduction region"), can include all those parts of the earth where induced currents flow. However the $\mathbf{E}(\mathbf{r})$ of eq. 14 is the solution of Maxwell's equation:

$$\text{curl } \mathbf{E}(\mathbf{r}) + \frac{\partial \mathbf{B}_T(\mathbf{r})}{\partial t} = 0 \quad (15)$$

where $\mathbf{B}_T(\mathbf{r})$ is the total magnetic field at point \mathbf{r} and $\partial/\partial t$ is the time derivative, and it is in specifying this equation for $\mathbf{E}(\mathbf{r})$ that it is possible to be more restrictive. An "induction region" may be defined as including only that space in which the presence of non-zero electrical conductivity contributes significantly through eq. 15 to magnetic variations as measured at an observing site. By definition, space outside the induction region is not important from the point of view of eq. 15, and may be excluded from computations pertaining to it. Because two different equations are involved, eq. 14 and 15, the conduction and induction regions are specified quite distinctly (though in practice they may of course overlap).

The condition requiring spatial uniformity of the primary field may now be relaxed to require spatial uniformity only within the induction region: by definition, induction outside the induction region is not important.

5. Discussion

Starting from the simplest general premises, it has first been shown that Parkinson's linear relation is not a general result for magnetic variations, but results

from either of two possible special conditions. The more reasonable of these two conditions is that the induction matrix for the site in question is simple degenerate, or nearly so, and the extent to which the matrix departs from this condition will in general cause departures of variations from the Parkinson plane. The departures of data from Parkinson's relation follow from consideration of the basic induction tensor, the elements of which are determined by the electrical conductivity structure of the earth. The discussion so far has been for primary fields spatially uniform, or at any rate spatially uniform within the induction region.

It has then been noted that another cause of departures of data from Parkinson's relation could be non-uniformity of the primary fields over the induction region. This would be most serious in cases of "current channelling", where current induced remotely is channelled near to an observing station, there to influence the magnetic variations recorded. Current channelling by itself will not necessarily cause observed data to depart from Parkinson's relation, but if it is on a large enough scale it is quite conceivable that the variations which cause the remotely induced current will be different from those measured locally at the observing site. Then the variation fields of the channelled currents will not necessarily correlate with the local primary fields so that observing sites in regions of extensive current channelling may be characterized by unusually poor fits to Parkinson's relation.

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