A Comparative Study of Explicit Differential Operators on Arbitrary Grids

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July 20, 1999

Abstract

We compare explicit differential operators for unstructured grids and their accuracy with the aim of solving time-dependent partial differential equations in geophysical applications. As many problems suggest the use of staggered grids we investigate different schemes for the calculation of space derivatives on two separate grids. The differential operators are explicit and local in the sense that they use only information of the function in their nearest neighbourhood, so that no matrix inversion is necessary. This makes this approach well suited for parallelization. Differential weights are obtained either with the finite-volume method or using natural neighbour coordinates. Unstructured grids have advantages concerning the simulation of complex geometries and boundaries. Our results show that while in general triangular (hexagonal) grids perform worse than standard finite-difference approaches, the effects of grid irregularities on the accuracy of the space derivatives are comparably small for realistic grids. This suggests that such a finite-difference-like approach to unstructured grids may be an alternative to other irregular grid methods such as the finite-element technique.

1 Introduction

Numerical simulation of wave propagation in general heterogeneous 2-D and 3-D media is becoming an increasingly important tool in understanding the wavefield recorded on all scales ranging from laboratory measurements and reservoir scales
to regional and global seismology. So far, numerical solutions to the wave equation have been dominated by (quasi-)regular grid methods such as the finite-difference (FD) method (e.g. Kelly et al. 1976) or the pseudospectral (PS) method (e.g. Fornberg 1987, 1988; Tessmer, Kosloff & Behle 1992). The numerical techniques capable of handling arbitrary grids such as the finite-element (FE) method (e.g. Smith 1975; Marfurt 1984), the finite-volume (FV) method (Dormy & Tarantola 1995), or the spectral element (SE) method (e.g. Padovani et al. 1994), had far less attention, probably because their implementation is more involved than regular grid methods. But methods with single domains and (quasi-)regular grids have their limitations. For example, when media are to be simulated with large velocity contrasts, then parts of the model are oversampled, because - for stability reasons - the grid has to be adjusted to the smallest velocities resulting in finer gridding. Furthermore, the accurate implementation of boundary conditions often requires denser gridding near the boundaries than within the medium. Multidomain regular grids are possible (Jastram & Tessmer 1994; Falk, Tessmer & Gajewski 1996), yet the implementation usually becomes far more difficult and less flexible. Another difficulty arises when one attempts to solve problems which suggest the use of curvilinear coordinates (e.g. cylindrical or spherical problems) as shown in Figure 1a. Regular gridding of cylindrical coordinates leads to decreasing grid spacing toward the axis \( r = 0 \) at the grid center, which in time-dependent problems requires unrealistically small time steps. Alternatively, when using irregular grids (e.g. Zhang & Tielin 1999; Zhang 1997), the problems can often be solved using the cartesian equations (Figure 1b) and the boundary conditions can be applied on the curved boundary. This is not possible with regular grid methods because the blocky nature of the curved boundary described by rectangles leads to strong artifacts.

In this paper we discuss different explicit differential operators on arbitrary grids and compare their accuracy with standard FD operators on rectangular grids and operators used for regular hexagonal grids, introduced by Magnier, Mora & Tarantola (1994). Magnier et al. propose their technique on hexagonal grids as an alternative to the classical Cartesian finite-difference approach and claim, that their solutions are at least as precise as classical staggered Cartesian finite-difference results. Our results suggest that the accuracy of the space derivatives on hexagonal grids are considerably worse than an equivalent FD-approach. In this study we are particularly interested in the question, how grid irregularity influences the accuracy of the space derivatives compared to schemes on regular grids with equivalent grid densities. While Dormy & Tarantola (1995) state that grid irregularity has only small effects, they did not perform a thorough quantitative analysis. In this study we quantitatively compare the accuracy of the numerical derivatives for harmonic trigonometric test functions as a function of grid (i.e. average triangle) quality.
2 Staggered Grid Schemes

Many time-dependent problems in geophysics suggest the use of staggered grid schemes as the governing equations contain only first derivatives. For example, the elastic wave equation can be written as a first order system (see Appendix A), suggesting that the velocities be defined on one grid (primary grid) and the stresses on another (secondary grid). This concept was first used for wave propagation by Virieux (1984, 1986). This so-called grid splitting or grid staggering is mainly caused by the definition of the derivative operators, as the values of the derivatives are located halfway between the function nodes. Staggered grids generally result in improved accuracy compared to non-staggered grids with all fields defined at the same locations, which is due to the antisymmetry of the differential operator.

For irregular grids the derivative values cannot in general be defined right in between two function values at nodes are not aligned along any coordinate axes. So the question of positioning the secondary grid with respect to the primary grid to define derivatives is non-trivial. Small sections of a standard rectangular staggered grid and an irregular staggered grid are shown in Figure 2. In the irregular case the velocities are defined on a primary grid. After Delaunay triangulation (e.g. Watson 1981; Sibson 1981; Devijver & Dekezel 1983), a method developed in the field of computational geometry, the secondary grid for the components of the stress tensor is obtained by using the triangle centres of the primary (velocity) grid. Note that all stress elements are defined at the same grid points (non-staggered). This would allow modeling of general material anisotropy without the need for additional interpolations which decrease the overall accuracy (Igel et al. 1995).

3 Irregular Grid Generation and Grid Quality

To test the differential operators we create irregular grids by randomly distorting perfect hexagonal grids. In contrast to regular grids, the coordinates of all grid points have to be known and all space-dependent fields are stored as vectors instead of two-dimensional arrays as is the case for regular 2-D grids. Therefore, the problem of defining neighbouring nodes on arbitrary grids becomes non-trivial. In order to navigate through irregular grids Delaunay triangulation is one of the most important tools. The primary grid is triangulated in a way to optimize the quality of the triangles. In other words the triangles are as equilateral as possible. The triangle quality $q$ can be written as

$$q = \frac{4\sqrt{3}A}{a^2 + b^2 + c^2}$$  \hspace{1cm} (1)
where A is the triangle area, and a, b, and c are the side lengths. This quality factor ranges from 0 to 1 with its maximum value for an equilateral triangle. Examples of triangular grids and their average triangle qualities are given in Figure 3. We expect that the accuracy of the numerical derivative calculated from the nearest neighbours depends on the quality of the involved triangles. Therefore in our investigation of local derivative operators on irregular grids the average triangle quality and the worst triangle quality will be important parameters. In this study we intend to investigate the accuracy of derivative operators for harmonic functions on irregular grids. To understand the influence of the irregularity on the accuracy we will always use a perfect triangular grid and a rectangular standard finite difference grid with the same grid density as reference. A grid consisting of perfect triangles is a hexagonal grid. Magnier et al. (1994) introduced differential operators which allow the partial derivative of fields to be evaluated at the center of the triangle. For wave propagation problems this leads to a staggered scheme where the function values are defined at the vertices of the triangle (primary grid) and the derivatives at the triangle centers (secondary grid). To ensure a reasonable comparison of results of different grids we use equivalent average node densities for each grid, regular or irregular.

4 Difference Weights on Arbitrary Grids

We investigate three different ways to calculate partial derivatives, the natural neighbour derivative (NND), introduced to geophysics by Sambridge et al. (1995) and Braun et al. (1995), the finite volume method using natural neighbours (FVN) (Dormy 1995) and the finite volume method using only three neighbours (FV3). In general, the derivative $\partial_i f(x_0)$ of the scalar function $f(x)$ will be obtained by calculating the sum over all neighbouring values $f(x_j)$ weighted by some value $w_j$ according to

$$\partial_i f(x_0) \approx \sum_{j=1}^{N} w_j f(x_j),$$

where $N$ is the number of neighbouring points (see Figure 4).

4.1 Natural Neighbour Weights

As mentioned above, the Delauney triangulation is the key to operate on irregular grids. A set of arbitrary points is triangulated by linking adjacent grid points so that optimal quality triangles result. A secondary grid is obtained by putting grid points into the centers of these triangles as shown in Figure 3. The goal is to evaluate partial derivatives on one grid knowing the function on the other. To achieve this we use the concept of natural neighbours (e.g. Sambridge et al. 1995; Watson 1985, 1992). Sambridge’s method of natural neighbour coordinates
determines the neighbouring points of each node, for which the partial derivative has to be evaluated. These neighbouring points are uniquely determined through the concept of Voronoi cells (see Sambridge et al. 1995; Fortune 1992). Once we have a list of neighbours and their coordinates differential weights can be calculated by simply using the derivative of the interpolation weights. The interpolation weights are obtained by the relative contribution of the neighbouring Voronoi cells (Sambridge et al. 1995). The formal expression for the calculation of natural-neighbour-weights are outlined in Appendix B.1.

4.2 Finite Volume Weights using all Neighbours

An alternative to this approach is the use of finite-volume weights (Dormy & Tarantola 1995). The FV method is based on a discretization of Gauss’ divergence theorem. For the calculation of the differential weights we again use the natural neighbours as introduced before and connect them to form a hull (FV cell) around the point where the derivative is to be evaluated. The length of the cell sides and the components of the corresponding normal vectors are then determined. Finally we sum over all natural neighbour values, weighted by the lengths of the adjacent cell sides and the corresponding components of the normal vectors.

The formal expression for the calculation of FV weights using all natural neighbours is given in Appendix B.2. Gauss’ theorem implies that the derivative is assumed constant within the cell and is independent of the location of the point where the derivative is to be evaluated.

4.3 Finite Volume Weights using three Neighbours

This method is identical to the FV method described above except the number of neighbours used for calculating the derivative is limited to three. As shown by Magnier et al. (1994) it is sufficient to use only three points for calculating spatial derivatives on a 2-D grid or four points in the 3-D case. In general, the authors call an N-dimensional grid using \( N + 1 \) points to compute derivatives a minimal grid.

While the previous methods of determining natural neighbours on irregular grids used the Delaunay-triangulation after inserting a secondary grid point, we face the problem, that we do not know which three of the natural neighbours will lead to the most accurate result. Therefore we use another method of finding the best three natural neighbours. As mentioned before, we start initializing a primary grid where the velocities are defined and obtain a triangular mesh by using the Delaunay algorithm. If we insert a secondary point into the existing primary grid, this point will be located inside of one Delaunay triangle of the primary grid. To calculate the spatial derivative at that secondary point using
only three neighbours, the best choices will be the three primary grid points that are forming the triangle in which the secondary point is located (see Figure 3). A major problem when operating with arbitrary grids is to know which triangle contains a given point. For an increasing number of primary grid points it is becoming increasingly inefficient to search through all triangles. Therefore we use the very efficient walking triangle algorithm as referred to in previous work (Sambridge et al. 1995; Lawson 1977), which quickly finds the triangle containing any given secondary point. Once we have determined the list of the three best neighbours, we can use the FV method for calculating the derivative. But the summation is limited to the three best neighbours.

4.4 Reference Cases

We compare the accuracy of the three different irregular grid methods, with two different regular grid techniques, a rectangular standard FD grid and a regular hexagonal grid. As the comparison of grids with identical grid spacing is impossible - due to varying sizes and shapes of irregular grid cells - we are using the average density of nodes as a measure to compare the accuracy of operators on equally spaced regular and irregular grids. We will compare different methods by the (average) number of grid points per wavelength \( \lambda \), with which the test function is sampled.

4.4.1 FD on Regular Grids

The FD grid used for the comparison consists of square shaped grid cells which leads to equal grid spacing in the \( x \)- and \( y \)-dimensions. We also use a staggered scheme as shown in Figure 2. The FD operator is second order in \( \Delta x \). The derivatives on the rectangular grid are calculated by using only two neighbouring points in each direction. As the numerical derivative is defined between the two function values, numerical and analytical results of the same location are compared in this case. The derivatives are simply computed by

\[
\frac{\partial_x f(x_0)}{\Delta x} \approx \frac{f_2 - f_1}{\Delta x},
\]

\[
\frac{\partial_y f(x_0)}{\Delta y} \approx \frac{f_4 - f_3}{\Delta y},
\]

where \( \Delta x \) is the grid spacing and \( f_i(i = 1, ..., 4) \) are the surrounding function values (see Figure 5a).

4.4.2 Hexagonal Minimal Grids

The hexagonal grid is equal to a perfectly shaped triangular grid. Therefore the derivative operator for the staggered scheme uses the three nodes forming
the primary grid cell (an equilateral triangle). In this case the weights can be calculated explicitly as shown by Magnier et al. (1994). The derivatives are computed by

\[ \partial_x f(x_0) \approx \frac{f_2 - f_1}{\Delta x} \]
\[ \partial_z f(x_0) \approx \frac{f_1 + f_2 - 2f_3}{\Delta x \sqrt{3}} \]

where \( \Delta x \) is the side length of the triangle and \( f_i(i = 1, 2, 3) \) are the surrounding function values (see Figure 5b). The derivative values are defined in between the corresponding function values, i.e. \( \partial_x f(x_0) \) is defined at location \( p_x \) and \( \partial_z f(x_0) \) at \( p_z \). The analytical derivative value of the test function is defined at the triangle center \( x_0 \), which is the equivalent location of the grid point on the secondary grid. Therefore analytical and numerical derivatives are located at different points in contrast to rectangular FD schemes.

5 Accuracy of Space Derivatives

On irregular grids an analytical analysis of the accuracy of derivatives is not possible. To determine the accuracy of the derivative operators we therefore use a two-dimensional sinusoidal test function \( f(x) = \sin(\pi x)\sin(\pi z) \) as shown in Figure 6. The function is defined in the interval \(-1 \leq x \leq 1 \) and \(-1 \leq z \leq 1 \) and remains unchanged for all test grids. If we discretize this test function using a staggered grid as described above, we have discrete function values defined on the primary (e.g. velocity) nodes. At the secondary (e.g. stress) nodes of the staggered grid we can numerically calculate space derivatives by applying the NND, FV3 or FVN method. We can also compute the derivative values of the test function at all secondary nodes analytically. To compare numerical and analytical results we determine the difference of numerical and analytical derivative values at each secondary grid point and calculate a mean absolute error \( \epsilon \), given by

\[ \epsilon = \frac{1}{n_s} \sum_{i=1}^{n_s} \left| \frac{\partial f_{x,z} - \partial f_{x,z}}{\pi} \right| \]

where \( n_s \) is the number of secondary points, at which numerical derivative values \( (\partial f_{x,z}) \) and analytical derivative values \( (\partial f_{x,z}) \) are compared. The division by \( \pi \) normalizes the error with respect to the maximum analytical derivative value. Sampling the test function with grids of varying node densities ranging from 6 to 100 grid points per wavelength \( \lambda \) and calculating the corresponding mean absolute error leads to the error-curves shown in Figure 7. To avoid effects due to the edges of the grid the summation is limited to secondary grid points with
coordinates in the interval \(-0.7 \leq x \leq 0.7\) and \(-0.7 \leq z \leq 0.7\). After computing these error values for nine different grids with grid-cells of varying average triangle qualities \((1.000 \geq q \geq 0.842)\), we obtain a series of nine graphs (see Figure 7). In realistic applications (e.g. a cylindrical problem as shown in Figure 1b) the average grid quality was observed to be around \(q \approx 0.93\). As references we also show error-curves obtained by using a rectangular standard FD grid and a hexagonal grid (Magnier-grid). Note that even for the hexagonal grid consisting of equilateral triangles the accuracy of the \(z\)- and especially of the \(x\)-derivatives is considerably worse than for the rectangular FD grid. Increasing distortion of the hexagonal grid in general leads to decreasing accuracy of the space derivatives. Particularly the accuracy of the \(z\)-derivative is strongly affected by grid irregularity, whereas the accuracy of the \(x\)-derivative does not change remarkably. The diagrams also show that the difference in the accuracy of \(x\)- and \(z\)-derivatives (as clearly apparent for the hexagonal case) decreases for increasing grid irregularity, i.e. for highly irregular grids there is no difference between the accuracy of derivatives in \(x\)- and \(z\)-direction.

Comparing the three described techniques of calculating space derivatives, the NND method provides the most accurate results of the numerical derivative values. For slightly distorted hexagonal grids this method even leads to higher accuracy of the \(x\)-derivative (see zoomed section in Figure 7) as Magnier’s method applied on perfect hexagonal grids, provided that the orientation of the triangles is chosen as shown in Figure 3. Surprisingly, the accuracy obtained by the FV3 method is comparable to NND, though we only use three neighbouring points in the computation of space derivatives. The FVN method is less precise than the other two, but does not show a considerable difference between the accuracy of \(x\)- and \(z\)-derivatives.

A more direct comparison between the three techniques is given in Figure 8, as error curves of different methods are shown in a single diagram. Accuracies of \(x\)- and \(z\)-derivatives are shown for four different grids with decreasing average triangle qualities of \(q = 1.000, q = 0.988, q = 0.930\) and \(q = 0.842\). For the hexagonal grid \((q = 1.000)\) all methods converge to a single line and provide the same results as Magnier’s method. Again accuracy of the space derivatives is decreasing as the average triangle quality is decreasing. In these plots it is clearly shown, that NND always provides more accurate results for computing numerical derivatives than the other methods do, independent of grid irregularity. We can also see, that we do not lose very much accuracy when using the FV3 method with respect to NND. Again note the considerable difference between the accuracies obtained by using the FD grid and the hexagonal Magnier-grid. The influence of grid irregularity on the \(x\)-derivative is comparatively small.
6 Discussion

The investigation of derivative operators on arbitrary triangular grids has shown, that the accuracy of space derivatives is dependent on the average quality of the grid cells. A remarkable result is, that even for undistorted hexagonal grids consisting of equilateral triangles \((q = 1)\) the derivatives are considerably less accurate than for rectangular FD grids. This effect can be explained by the fact that numerical and analytical derivatives in the staggered hexagonal scheme are not defined at (or interpolated to) the same location as it is the case in the staggered FD scheme (see Figure 5). The effect of deviating locations of derivative values does not appear for plane waves travelling along the \(x\)- or \(z\)-axis, i.e. the \(z\)-derivative of a plane wave travelling in \(x\)-direction is constant in \(z\)-direction and the \(x\)-derivative of a plane wave travelling in \(z\)-direction is constant in \(x\)-direction. With respect to hexagonal grids, the introduced derivative operators on arbitrary grids yield comparable accuracy in \(x\)-direction and are in contrast to the \(z\)-direction relatively less affected by increasing grid irregularity. Whereas the accuracy of \(z\)-derivatives is high for hexagonal grids (assuming a triangle orientation as shown in Figure 3), the accuracy of the \(x\)-derivative is rather low. Therefore additional distortion of a hexagonal grid, does rather affect the precise \(z\)-derivatives than the less accurate \(x\)-derivatives. For highly irregular grids without any structure the difference of accuracies of \(x\) and \(z\)-derivatives finally vanishes.

Eventually, when using explicit local differential operators on arbitrary grids as introduced in this work, the NND method provides the most accurate results as the location of the secondary grid point with respect to the surrounding neighbours is considered in the computation algorithm. Contrary, the FVN and the FV3 method both assume the derivative to be constant within the entire FV-cell, which leads to low accuracy especially when using large cells of the FVN method, i.e. cells, that are defined by more and therefore more distant neighbours. The FV3 method is very similar to the approach of Magnier et al. as it also uses a minimal number of neighbours defining a plane of linear interpolation, with the extension that the triangle can be arbitrary shaped and oriented. As the FV3 algorithm only uses three neighbours to compute space derivatives on the staggered grid it is very fast, which offers the possibility to use denser grids and potentially achieve higher accuracy than NND, without considerably increasing computation time.

7 Conclusion

Different local operators can be designed to compute spatial derivatives on an arbitrary grid. We have shown, how the grid irregularity affects the accuracy of the differential operators. The operator based on the natural-neighbour-method
(NND) provides most accurate results independent of grid irregularity. Problems of difference operators on hexagonal grids as introduced in previous work (Magnier \textit{et al.} 1994) have been pointed out and all results were compared to standard finite-difference methods on rectangular grids with equivalent node densities. The important property of all operators described is, that they are local in the sense that they use only information of a function in the nearest neighbourhood of the point where the derivative is to be calculated. A potential possibility of these local operators is their application in large scale simulation problems as they are well suited for parallelization. In this regard it is also important that the insignificantly less accurate FV3 method is a very fast algorithm for computing space derivatives, especially with the aim of extending the methods to 3-D. Additional research is required to test the operators in simulation algorithms of elastic wave propagation on arbitrary grids. The implementation of boundary conditions with these operators also deserves further study.

**Acknowledgments**

Thanks to Gunnar Jahnke, Helmut Gebrane and Hamish Macintyre for critical comments and helpful discussions to improve this paper.
References


A Elastic wave equation as first order system

The equations of motion for elastic wave propagation are

\[
\begin{align*}
\rho \partial_t v_i &= \nabla_j (\sigma_{ij} + M_{ij}) + f_i \\
\partial_t \sigma_{ij} &= c_{ijkl} \partial_t \epsilon_{ij} \\
\partial_t \epsilon_{ij} &= \frac{1}{2} (\partial_i v_j + \partial_j v_i)
\end{align*}
\]

where \( v_i \) are the components of velocity, \( \sigma_{ij} \) and \( M_{ij} \) are the elements of stress and moment tensor, respectively, \( \rho \) is the mass density, \( \epsilon_{ij} \) are the elements of the deformation tensor, \( c_{ijkl} \) are elastic stiffnesses, and \( f_i \) are volumetric forces. As derivatives are discretized using centered classical finite-differences, the equations lead to a staggered grid as shown in Figure 2.

B Derivative Weights

B.1 Natural Neighbour Weights

A detailed description of the calculation of the natural neighbour weights is given by Sambridge et al. (1995). The method was originally developed for solving interpolation problems of irregularly distributed data. The method is based on the theory of Voronoi cells and the Delaunay triangulation algorithm. The interpolation weights are computed by the relative contribution of second-order Voronoi cells and is formally given by

\[
f(x_0) \approx \frac{1}{A} \sum_{j=1}^{N} a_j(x_0) f(x_j)
\]

where \( N \) is the number of considered neighbours, \( a_j \) are the areas of second-order Voronoi cells and \( A \) is the total area of the Voronoi cell about the point \( x_0 \) which is

\[
A = \sum_{j=1}^{N} a_j(x_0) .
\]

Extending this method to the problem of calculating derivatives is easily done by differentiating equation (4) with respect to the different directions, leading to
\[
\partial_i f(x_0) \approx \frac{1}{A} \sum_{j=1}^{N} \partial_i a_j(x_0) f(x_j) - \frac{1}{A^2} \partial_i A \sum_{j=1}^{N} a_j(x_0) f(x_j)
\]

(6)

with \((i = 1, 2)\) for the 2-D case. An extension to 3-D is straightforward and terms of areas or triangles can be easily replaced by volumes or tetrahedra (Braun et al. 1995).

### B.2 Finite Volume Weights

A detailed description of the calculation of finite volume weights is given by Dormy et al. (1995). The method is based on Gauss’ divergence theorem, which in the 2-D case can be written as

\[
\int_S dS \partial_i f = \int_L dL n_i f
\]

(7)

where \(f\) is a vector field, \(n_i\) the outward normal vector and \(L\) the boundary of the cell surface \(S\).

The discrete formulation of Gauss’ theorem leads to

\[
\Delta S \partial_i f \approx \sum_{j=1}^{N} \Delta L_j n_{j,i} f \Rightarrow \partial_i f \approx \frac{1}{S} \sum_{j=1}^{N} \Delta L_j n_{j,i} f.
\]

(8)

As the neighbouring points define the vertices of the finite volume cell each neighbour is weighted by the sum of half of the adjacent two cell sides multiplied by the corresponding normal vectors.
Figure 1: (a) Finite difference grid for modelling in cylindrical coordinates. Note the necessary depth-dependent grid spacing to comply with the stability criterion. (b) Triangular grid for modelling a cylindrical problem in cartesian coordinates. Note the approximately *equal* grid spacing throughout the section.

Figure 2: (a) Example of a staggered grid scheme for a standard FD grid with quadratic grid cells (e.g. Virieux 1986). (b) Example of a staggered grid scheme for an irregular grid with triangular grid cells.
Figure 3: Four different sections of grids with equal node density are shown. Increasing grid irregularity leads to decreasing average triangle quality $q$. Also note the appearance of very low-quality triangles in highly irregular grids. To test operator accuracies staggered (secondary) grid points are initialized in the center of gravity of each primary grid cell.
Figure 4: Calculation of derivatives on a 2-dimensional irregular grid using function values at neighbouring points \( N1, \ldots, N5 \).
Figure 5: (a) FD grid cell (staggered scheme) to compute $\partial_{x,z} f(x_0)$ at the center of the cell. Note that the numerical derivatives are defined at $x_0$. (b) Triangular grid cell (staggered scheme) of a hexagonal grid to compute $\partial_{x,z} f(x_0)$ at the center of the cell. Note that here the numerical value of $\partial_{x,z} f(x_0)$ is defined at (i.e. interpolated to) point $p_x$ and $\partial_z f(x_0)$ at point $p_z$. The analytical values of $\partial_{x,z} f(x_0)$ are evaluated at $x_0$, the location of the grid point on the secondary grid.

Figure 6: Two-dimensional sinusoidal function to test operator accuracy by comparing analytical derivatives with numerically calculated derivative values.
Figure 7: Mean absolute errors ε (as percentage of the maximum value π) of the numerically computed x- and z-derivatives of the test function are shown for the three different methods. Graphs are drawn for nine different grid qualities ranging from $0.842 \leq q \leq 1.000$. The accuracies obtained by the FD grid and the perfect hexagonal grid are also given as references (thick lines).
Figure 8: Errors of the numerically computed $x$-derivatives (a) and $z$-derivatives (b) of the test function are shown for four different grids with average triangle qualities $q = 1.000, q = 0.988, q = 0.930$ and $q = 0.842$. NND clearly is more accurate than FV3 or FVN. Note the decreasing accuracy with decreasing grid quality $q$. The accuracies obtained by the FD grid and the perfect hexagonal grid are also given as references.